Matrices and networks - Practice test 4

10 questions | 40 marks | 65 minutes

An original mixed chapter paper. Show calculations, state units and justify network decisions using the supplied information.

Equivalent correct methods earn the corresponding marks.

1. $A=\begin{pmatrix}3 & 1 & 0\\ 2 & 5 & 4\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 0 & 3\\ 4 & 1\end{pmatrix}$. [4 marks]

(a) State the orders of AB and BA. [2 marks]

Check the matching inside dimensions in each order.

Answer: AB: 2×2; BA: 3×3.

(b) Calculate $(AB)_{21}$ and $(BA)_{32}$. [2 marks]

$2(1)+5(0)+4(4)=18$.

$4(1)+1(5)=9$.

Answer: 18 and 9.

2. An order row is $Q=\begin{pmatrix}2 & 5\end{pmatrix}$ for bread and milk. Bread costs 4 dollars and milk 3 dollars. A student uses price column $\begin{pmatrix}3\\ 4\end{pmatrix}$. [4 marks]

(a) State the student’s result and the correct result. [2 marks]

Student: $2(3)+5(4)=26$.

Correct: $2(4)+5(3)=23$.

Answer: Student 26 dollars; correct 23 dollars.

(b) Explain the modelling error. [2 marks]

Dimensions match but categories do not.

Bread must use 4 and milk 3.

Answer: The price-column order does not match the quantity-column labels.

3. Let $A=\begin{pmatrix}1 & 3\\ 2 & 0\end{pmatrix}$ and $B=\begin{pmatrix}2 & 1\\ 0 & 4\end{pmatrix}$. [4 marks]

(a) Calculate AB. [2 marks]

Use a row and column for every entry.

Answer: $\begin{pmatrix}2 & 13\\ 4 & 2\end{pmatrix}$.

(b) Calculate BA and give a numerical counterexample to commutativity. [2 marks]

Reversing factors changes the row/column pairings.

A single unequal entry proves the matrices differ.

Answer: $\begin{pmatrix}4 & 6\\ 8 & 0\end{pmatrix}$; for example top-left $4\neq 2$.

4. Rows represent stores 1 and 2; columns represent products A and B. Opening stock $S=\begin{pmatrix}12 & 8\\ 5 & 14\end{pmatrix}$ is followed by two identical deliveries $D=\begin{pmatrix}3 & 2\\ 4 & 1\end{pmatrix}$ and sales $R=\begin{pmatrix}8 & 7\\ 6 & 9\end{pmatrix}$. [4 marks]

(a) Write and calculate the closing-stock expression. [2 marks]

Double each delivery entry, then add and subtract.

Answer: $S+2D-R=\begin{pmatrix}10 & 5\\ 7 & 7\end{pmatrix}$.

(b) If the first product costs 5 dollars and the second 2 dollars, find each store’s closing value. [2 marks]

Store 1: $10(5)+5(2)=60$.

Store 2: $7(5)+7(2)=49$.

Answer: 60 and 49 dollars.

5. An undirected graph has edges AB, BC, CD, DA and AC. [4 marks]

(a) Is any edge a bridge? Explain. [2 marks]

The perimeter is a cycle.

The diagonal also has an alternative path between its endpoints.

Answer: No.

(b) If AC and AB are both removed, is the remaining graph connected? [2 marks]

The remaining edges BC, CD, DA form the path B-C-D-A.

Every vertex is still reached.

Answer: Yes.

6. Travel times are A→B 3 min, B→A 8 min, B→C 5 min, C→B 4 min, A→C 10 min and C→A 7 min. [4 marks]

(a) Find the fastest A-to-C journey. [2 marks]

Via B takes $3 + 5 = 8$.

Direct travel takes 10.

Answer: A→B→C;8 min.

(b) Find the fastest C-to-A journey and explain why it need not reverse the first. [2 marks]

Via B takes $4 + 8 = 12$.

Direction-dependent weights change the optimum.

Answer: C→A;7 min.

7. Rows and columns of $N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ are A, B, C, with rows as starting vertices. [4 marks]

(a) Find $(N^3)_{CB}$. [2 marks]

N^2 row C is $(0, 1, 1)$.

Multiply by column B $(1, 0, 0)^T$ to get 0.

Answer: 0.

(b) Find $(N^3)_{CC}$ and list the corresponding walk. [2 marks]

The same N^2 row times column C gives 1.

Trace all three directed edges.

Answer: 1; C→A→B→C.

8. A directed acyclic graph has A→B 2, A→C 6, B→C 1, B→D 8, C→D 3, D→E 2 and C→E 10. [4 marks]

(a) Find a shortest A-to-E path. [2 marks]

Candidate totals are 12, 13, 11, 16 and 8.

The minimum is $2 + 1 + 3 + 2 = 8$.

Answer: A→B→C→D→E;8.

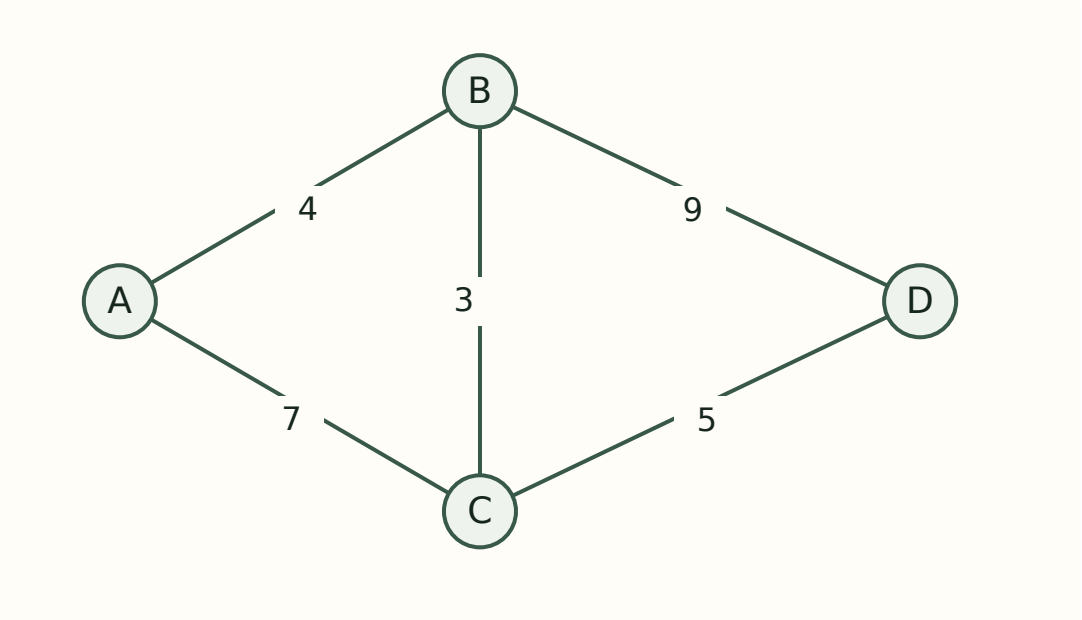
(b) Find a longest A-to-E path. [2 marks]

A→C→E gives $6 + 10 = 16$.

All other listed totals are lower.

Answer: A→C→E;16.

9. Connection costs are shown. [4 marks]



(a) Find a minimum spanning tree. [2 marks]

Use three edges connecting all four vertices.

The minimum total is $4 + 3 + 5 = 12$.

Answer: A-B 4, B-C 3, C-D 5; total 12.

(b) A new A-D edge costs 2. Find the new minimum tree. [2 marks]

The new edge links A and D cheaply.

Add B-C 3 and join the components with A-B 4.

Answer: A-D 2, B-C 3, A-B 4; total 9.

10. A directed network has S→A 7, S→B 5, A→B 3, A→T 4 and B→T 8. Capacities are units/hour. [4 marks]

(a) Find the capacity of the cut with source side {S, A} and sink side {B, T}. [2 marks]

Forward crossings are S→B 5, A→B 3, A→T 4.

$5 + 3 + 4 = 12$.

Answer: 12 units/hour.

(b) Show a feasible flow reaching 12 and hence prove the maximum. [2 marks]

A conserves $7 = 3 + 4$ and B conserves $5 + 3 = 8$.

The matching 12-capacity cut gives the upper bound.

Answer: SA 7, SB 5, AB 3, AT 4, BT 8; maximum 12.