Matrix models - Applications test 6

10 questions | 59 marks | 70 minutes

Linked models, inference and justified recommendations. Network matrices use rows=start and columns=destination; transition matrices use columns=current and rows=next unless stated otherwise.

Equivalent correct methods earn the corresponding marks.

1. A closed two-state matrix has rows $(0.9,b),(0.1,1-b)$. A current state (60,40) is predicted to become (62,38). [6 marks]

(a) Find b and check the resulting matrix. [3 marks]

$0.9(60)+40b=62$, so $b=8/40=0.2$.

Answer: $b=0.2$; rows $(0.9,0.2),(0.1,0.8)$ are column-stochastic.

(b) Find steady counts for a total of 300. [3 marks]

Balance gives $0.1a=0.2c$, so ratio 2:1.

Answer: (200,100).

2. A service-count adjacency matrix has rows $(0,k,1),(0,0,2),(1,0,0)$, ordered A,B,C, with k a nonnegative integer. [6 marks]

(a) If there are six two-step walks A to C, find k. [3 marks]

The (A,C) entry of the square is $2k$.

Solve $2k=6$.

Answer: $k=3$.

(b) Find the number of three-step A-to-C walks and explain the route. [3 marks]

The A row of the square is (1,0,6), whose dot product with column C is 1.

Answer: 1: A→C→A→C.

3. Three towns form the directed cycle A→B→C→A. [5 marks]

(a) Find which of $A_{AB}, (A^2)_{AB}, (A^3)_{AB}, (A^4)_{AB}$ are nonzero. [3 marks]

A returns to the same phase every three steps.

Answer: The first and fourth are 1; the other two are 0.

(b) Explain why a route-planning app must distinguish reachability from a required exact number of transfers. [2 marks]

For example one or four moves works; two or three does not.

Answer: B is reachable from A, but only certain exact lengths occur in this network.

4. A ranking rule counts direct wins and two-step wins. Teams A,B,C form the cycle A beats B, B beats C, C beats A. [5 marks]

(a) Show that the rule cannot select a unique winner if all direct wins get weight 1 and all two-step wins the same weight w. [3 marks]

Each team has one direct and one two-step relation.

Answer: Every team scores $1 + w$.

(b) Give two pieces of additional evidence that could improve a ranking model without merely inventing a tie-break. [2 marks]

The extra evidence and weighting rule must be transparent.

Answer: Examples: match margins, repeat-match results or independently specified strength-of-opposition information.

5. A 100-member population uses transition rows (0.9,0.4),(0.1,0.6). The next state is (60,40). [6 marks]

(a) Recover the current state. [4 marks]

Let current A be a. Then $0.9a + 0.4(100 - a) = 60$.

Thus $0.5a = 20$, so $a = 40$.

Answer: (40,60).

(b) Verify both destination counts. [2 marks]

Both equations agree with the observed forecast.

Answer: A:36+24=60; B:4+36=40.

6. Start at (100,0). Let T have rows (0.8,0.3),(0.2,0.7), and B rows (0.7,0.2),(0.3,0.8). [7 marks]

(a) Apply policies T, then B, then T. Write the product and final state. [4 marks]

The states are (80,20), then (60,40), then (60,40).

Answer: $TBTx_0 = (60, 40)$.

(b) Compare the final state for policies B, then T, then B. [3 marks]

The states are (70,30), then (65,35), then (52.5,47.5).

Answer: $BTBx_0 = (52.5, 47.5)$.

7. A three-state matrix has rows $(0.5,a,0),(0.5,1-2a,0.5),(0,a,0.5)$. [5 marks]

(a) Find the allowable range of a for a transition matrix. [2 marks]

Entries a must be nonnegative and $1-2a$ must be nonnegative. Columns already sum to 1.

Answer: $0 \le a \le 0.5$.

(b) Find a if (25,50,25) is steady. [3 marks]

First-state balance gives $25 = 0.5(25) + 50a$.

Solve $50a = 12.5$ and verify the other states.

Answer: $a = 0.25$.

8. A two-state policy keeps 85% of current A in A. The proportion of B moving to A is b. [7 marks]

(a) Choose b so the steady A proportion is 0.7. [4 marks]

Steady balance is $0.15(0.7) = b(0.3)$.

Thus $b = 0.105/0.3 = 0.35$.

Answer: $b = 0.35$.

(b) If b rises to 0.4, find the new steady A proportion and interpret. [3 marks]

A larger incoming B-to-A proportion raises the steady share of A, under unchanged assumptions.

Answer: $0.4/(0.15 + 0.4) = 8/11 \approx 0.7273$.

9. A model uses rows (0.8,0.3),(0.2,0.7), initially (200,80). The observed next state is (180,100). [5 marks]

(a) Find the one-period prediction and maximum absolute error. [3 marks]

A:160+24=184; B:40+56=96.

Answer: Prediction (184,96); maximum absolute error 4.

(b) The model's two-period prediction is exactly the later observation (176,104). Does this validate the model permanently? [2 marks]

Continue checking independent data and assumptions.

Answer: No; an exact match at one later time does not establish future constant behaviour.

10. Options F and S use transition rows (0.8,0.3),(0.2,0.7) and (0.98,0.03),(0.02,0.97), respectively. Start at (100,0). Per-period service cost is AUD 5 for A and AUD 1 for B. [7 marks]

(a) Compare expected costs using the distribution after three periods. [4 marks]

A counts are 65 and $60 + 40(0.95)^3 = 94.295$.

Costs are $5a + (100 - a) = 100 + 4a$.

Answer: F: AUD 360; S: AUD 477.18.

(b) Compare steady-state costs and explain why the preferred short-term option need not follow from equilibrium alone. [3 marks]

Both steady proportions are (0.6,0.4).

A fixed long-run split does not specify the speed of approach or transition-period cost.

Answer: Both steady costs are AUD 340, but their finite-period distributions differ.