Marking guide | Quadratics and polynomial graphs

Quadratics and polynomial graphs - Challenge and applications

10 questions | 54 marks | 75 minutes

A demanding mixed paper combining parameter reasoning, polynomial structure and constrained decisions.

Equivalent correct methods earn the corresponding marks.

1. Compare $A(x) = (x + 3)^2$ and $B(x) = x^2 + 9$. [4 marks]

(a) Find every x for which A(x)=B(x). [2 marks]

Expand A as $x^2 + 6x + 9$.

Equality reduces to 6x=0.

Answer: $x = 0$

(b) Explain why one example of equality does not prove these expressions are identical. [2 marks]

For example at x=1, A=16 but B=10.

Answer: An identity must hold for every allowed x; these agree only at zero.

2. A quadratic has vertex (2,-3) and passes through (-1,15). [5 marks]

(a) Find the rule in vertex and expanded forms. [3 marks]

Substitute (-1,15): 15=9a-3, so a=2.

Expand the square.

Answer: $y = 2(x - 2)^2 - 3 = 2x^2 - 8x + 5$.

(b) Find the exact real roots. [2 marks]

Set $(x - 2)^2 = 3/2$.

Take both square-root branches.

Answer: $x = 2 \pm \frac{\sqrt{6}}{2}$

3. Consider $x^2 - 4x = 0$ and $x^2 - 4 = 0$. [4 marks]

(a) Solve both equations and explain why their solution sets differ. [2 marks]

The first factors as x(x-4).

The second factors as (x-2)(x+2).

Answer: First: 0,4; second: -2,2.

(b) A student divides the first equation by x. Explain precisely what information is lost. [2 marks]

Division by x requires x nonzero.

The original equation is satisfied when x=0.

Answer: The solution x=0.

4. The equation $ax^2 + 4x + 1 = 0$ is to have two distinct positive roots. [6 marks]

(a) Use the root sum and product to show no such real a exists. [3 marks]

If a=0 the equation is linear and has only one root, so assume a is nonzero.

Positive root product requires 1/a>0, hence a>0.

But then root sum -4/a is negative, so the two roots cannot both be positive.

Answer: No real a gives two positive roots.

(b) Find the values of a for which it has two distinct negative real roots. [3 marks]

For two negative roots, positive product and negative sum require a>0.

Two distinct real roots require 16-4a>0.

Combine the conditions.

Answer: $0 < a < 4$.

5. A monic cubic $p(x) = x^3 + ax^2 + bx - 8$ has factors x-2 and x+1. [6 marks]

(a) Find a and b. [4 marks]

p(2)=0 gives 2a+b=0.

p(-1)=0 gives a-b=9.

Solve the two equations.

Answer: $a = 3$, $b = -6$.

(b) Find the remaining factor and all roots. [2 marks]

The full factorisation is $(x - 2)(x + 1)(x + 4)$.

Expansion confirms the coefficients and constant.

Answer: Remaining factor x+4; roots -4,-1,2.

6. Consider $g(x) = x^4 - kx^2 + 4$. [6 marks]

(a) Find k for which the quadratic in u=x squared has a repeated root. [2 marks]

The u-equation is $u^2 - ku + 4 = 0$.

Set its discriminant $k^2 - 16 = 0$.

Answer: $k = 4$ or $k = -4$.

(b) For each value, state every real x-root and its multiplicity. [4 marks]

At k=4, $g = (x^2 - 2)^2$.

At k=-4, $g = (x^2 + 2)^2$, which is strictly positive for real x.

Answer: k=4: $x = \pm\sqrt{2}$, each double. k=-4: no real roots.

7. The rule $f(x) = -(x - 5)^2 + 9$ is restricted to $1 \le x \le 4$. [4 marks]

(a) Find its range on that domain. [3 marks]

The function is increasing to the left of x=5.

Evaluate f(1)=-7 and f(4)=8; endpoints are included.

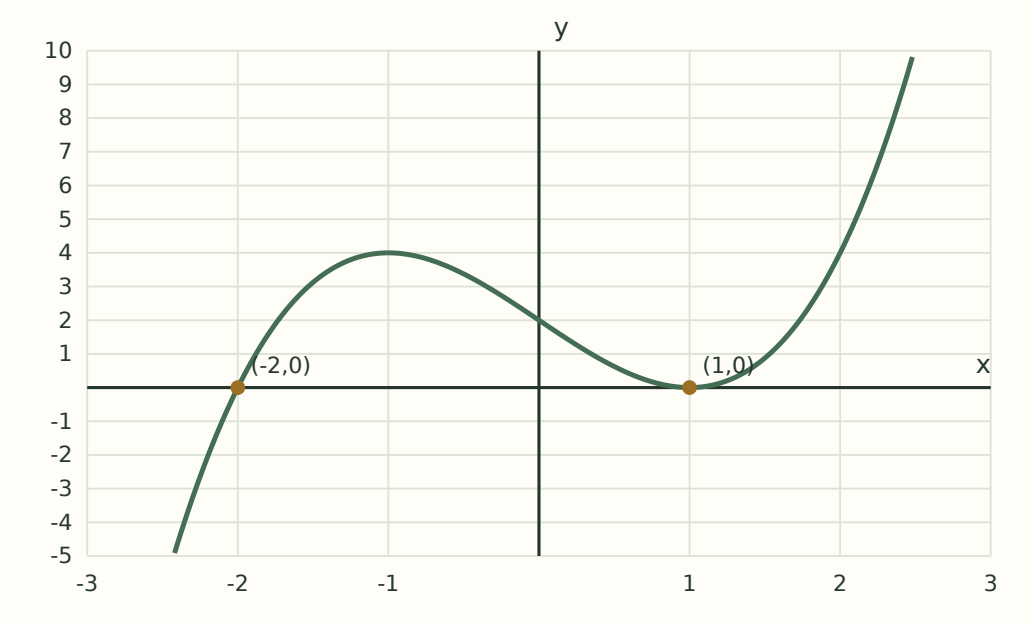
Answer: $-7 \le y \le 8$.

(b) Explain why 9 is not an attained output on this domain. [1 mark]

The unrestricted vertex is not part of the restricted graph.

Answer: It requires x=5, which is excluded.

8. The graph shown has rule $p(x) = (x+2)(x-1)^2$. [6 marks]



(a) Explain its behaviour at each x-intercept and state p(0). [3 marks]

Use odd/even factor multiplicities.

Substitute zero.

Answer: Crosses at -2; touches at 1; p(0)=2.

(b) Find every intersection with the horizontal line y=2 exactly. [3 marks]

Expand p as $x^3 - 3x + 2$.

Set p=2, so $x(x^2 - 3) = 0$.

Solve all factors.

Answer: $x = -\sqrt{3}, 0, \sqrt{3}$; all points have y=2.

9. Two positive numbers differ by 2 and their product is 35. [6 marks]

(a) Find the numbers using a quadratic equation. [3 marks]

Let the smaller be x>0: x(x+2)=35.

Factor $x^2 + 2x - 35 = (x + 7)(x - 5)$.

Reject x=-7.

Answer: 5 and 7.

(b) Form a quadratic equation with integer coefficients whose roots are the reciprocals of these numbers. [3 marks]

The reciprocal sum is 1/5+1/7=12/35.

The reciprocal product is 1/35; multiply the monic equation by 35.

Answer: $35t^2 - 12t + 1 = 0$.

10. A seller models the number of items sold as $N = 26 - 2p$, where p is a whole-dollar price with $0 < p < 13$. Fixed daily costs are 60 dollars. [7 marks]

(a) Write profit as a quadratic and find every allowed price giving positive profit. [4 marks]

Profit is revenue pN minus 60.

Factor $P = -2(p - 3)(p - 10)$.

Positive profit requires 3<p<10; retain whole-dollar values.

Answer: $P = -2p^2 + 26p - 60$; prices $4, 5, 6, 7, 8, 9$.

(b) Find the best allowed price or prices and the predicted profit. [3 marks]

The real-valued vertex is p=6.5, between two allowed integers.

Check p=6 and 7; symmetry gives the same value.

Values farther from the vertex give less profit.

Answer: $p = 6$ or $p = 7$, each giving 24 dollars profit.