Introduction to calculus - Topic test 2

10 questions | 52 marks | 65 minutes

Move between numerical data, derivative information and polynomial graphs, then interpret models.

Equivalent correct methods earn the corresponding marks.

1. A position table gives s(0)=3, s(1)=5, s(2)=11 and s(3)=21 metres. [5 marks]

Position measurements

t(s)	0	1	2	3
s(m)	3	5	11	21

(a) Find the average velocities on [0,1], [1,2] and [2,3]. [3 marks]

Subtract consecutive positions; each interval is one second.

Answer: 2, 6 and 10m/s.

(b) Use [1,2] to estimate velocity near 1.5s. Is it necessarily exact? [2 marks]

A secant slope approximates a local rate; a function or limiting evidence is needed for exactness.

Answer: About 6m/s; not necessarily exact.

2. Use $f(x) = x^3$ at x=1. [4 marks]

(a) Simplify its difference quotient. [3 marks]

Expand $(1+h)^3 - 1 = 3h + 3h^2 + h^3$ and divide by h≠0.

Answer: $3 + 3h + h^2$

(b) Find the instantaneous gradient. [1 mark]

Let h tend to 0 in the simplified quotient.

Answer: 3

3. A quadratic is $f(x) = ax^2 + bx + c$, with $f'(0) = -4$, $f'(2) = 8$ and $f(0) = 5$. [4 marks]

(a) Find a and b. [3 marks]

$f' = 2ax + b$; at 0, b=−4; at 2, 4a−4=8, giving a=3.

Answer: a=3, b=−4.

(b) Write the full function. [1 mark]

The original value at 0 gives c=5.

Answer: $f(x) = 3x^2 - 4x + 5$

4. For $y = x^3$, [6 marks]

(a) find the normal at x=1. [3 marks]

The point is (1,1); tangent gradient is 3; normal gradient is −1/3.

Answer: $y - 1 = -\frac{1}{3}(x - 1)$

(b) find all points where the tangent gradient is 3. [3 marks]

$3x^2 = 3$ gives x=±1; substitute into x^3.

Answer: (−1,−1) and (1,1).

5. Compare $f(x) = x^3$ and $g(x) = -x^4$ at x=0. [5 marks]

(a) Show that each has a stationary point there. [2 marks]

$f' = 3x^2$ and $g' = -4x^3$.

Answer: Both derivatives are 0 at 0.

(b) Classify each using signs. [3 marks]

For f, slopes are positive on both sides. For g, slopes change positive to negative.

Answer: f: neither; g: local maximum.

6. Analyse $f(x) = x^3 - 6x^2 + 9x$. [7 marks]

(a) Find all intercepts and stationary coordinates. [4 marks]

$f = x(x-3)^2$; $f' = 3(x-1)(x-3)$.

Evaluate f at 1 and 3.

Answer: Intercepts (0,0),(3,0); stationary (1,4),(3,0).

(b) Give monotonic intervals and stationary types. [3 marks]

Derivative signs are +,−,+.

Answer: Increasing x<1 or x>3, decreasing 1<x<3; maximum (1,4), minimum (3,0).

7. $s(t) = t^3 - 3t^2$ metres on $0 \le t \le 4$ seconds. [6 marks]

(a) Find rest times and the interior change of direction. [3 marks]

$v = 3t(t-2)$. It is negative on (0,2), positive on (2,4).

Answer: At rest at 0s and 2s; interior turn at 2s.

(b) Find displacement and total distance. [3 marks]

Positions at 0,2,4 are 0,−4,16; distance 4+20=24.

Answer: Displacement 16m; distance 24m.

8. A garden against a wall has 48m of fencing for the other three sides. [5 marks]

(a) Form its area in terms of the perpendicular side x. [2 marks]

The parallel side is 48−2x.

Answer: $A = x(48 - 2x)$ for $0 < x < 24$.

(b) Find the greatest area. [3 marks]

$A' = 48 - 4x$ is 0 at 12, changing + to −; A(12)=288.

Answer: 288m²; sides 12m,12m,24m.

9. Cost is $C(x) = 0.25x^2 - 5x + 90$ on $0 \le x \le 30$. [5 marks]

(a) Find and classify its stationary point. [3 marks]

$C' = 0.5x - 5 = 0$ gives 10; signs − to +.

Answer: Minimum at x=10, cost 65.

(b) Check that it is the global minimum on this domain. [2 marks]

$C(0) = 90, C(30) = 225 - 150 + 90 = 165.$

Both exceed 65.

Answer: Yes: endpoint costs are 90 and 165.

10. A quadratic has derivative $f'(x) = 2x - 4$ and $f(1) = 7$. [5 marks]

(a) Find f by comparing quadratic coefficients. [3 marks]

Write f=ax²+bx+c. Its derivative gives a=1,b=−4; f(1)=7 gives c=10.

Answer: $f(x) = x^2 - 4x + 10$

(b) Find its minimum value. [2 marks]

$f' = 0$ at 2, with − to +; f(2)=4−8+10=6.

Answer: 6 at x=2.