Introduction to calculus - Topic test 5

10 questions | 56 marks | 75 minutes

A harder synthesis paper with exact tangent conditions, parameters, weighted constraints and model evaluation.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = x^3 - 3x$. [4 marks]

(a) Find its average rate on $[-1,1]$. [2 marks]

$f(-1) = 2$, $f(1) = -2$; $(-2 - 2)/(1 - (-1)) = -2$.

Answer: -2

(b) Compare with its instantaneous rate at 0. [2 marks]

$f' = 3x^2 - 3$, so $f'(0) = -3$.

A secant uses two endpoints; a tangent uses local limiting behaviour.

Answer: -3; the average is not the tangent gradient at 0.

2. A quadratic has form $f(x) = x^2 + bx + c$. [6 marks]

(a) Find the difference quotient at input a and its limit. [3 marks]

$f(a + h) - f(a) = 2ah + h^2 + bh$.

Divide by nonzero h and let h tend to 0.

Answer: $2a + b + h$; limit $2a + b$.

(b) If $f'(3)=0$ and $f(1)=5$, find b and c. [3 marks]

$6+b=0$ gives $b=-6$. Then $1-6+c=5$ gives $c=10$.

Answer: $b=-6$, $c=10$.

3. Find tangents to $f(x) = x^3 - 3x$ parallel to $3x - y + 2 = 0$. [6 marks]

(a) Find the possible tangent inputs. [3 marks]

The line gradient is 3. Solve $3x^2 - 3 = 3$.

Answer: $x = \pm\sqrt{2}$

(b) Give both exact tangent equations. [3 marks]

At input a=±√2, f(a)=−a and slope=3.

Point-gradient form gives y+a=3(x−a), hence y=3x−4a.

Answer: $y = 3x - 4\sqrt{2}$ and $y = 3x + 4\sqrt{2}$

4. $f(x) = x^3 - 3kx$ has a stationary point at x=2. [5 marks]

(a) Find k and all stationary coordinates. [3 marks]

$f' = 3x^2 - 3k$; 12−3k=0 gives k=4.

Solve f′=3x²−12=0 and evaluate f.

Answer: k=4; (−2,16) and (2,−16).

(b) Classify them. [2 marks]

Derivative signs are +,−,+.

Answer: Maximum (−2,16); minimum (2,−16).

5. A model cost is $C(x) = 2x^2 - 16x + 50$ for $1 \le x \le 6$. [4 marks]

(a) Find its stationary value. [2 marks]

$C' = 4x - 16 = 0$ gives 4; C(4)=32−64+50=18.

Answer: 18 at x=4.

(b) Use all relevant candidates to justify the absolute minimum. [2 marks]

The stationary point is in the domain and both endpoint values are larger.

Answer: 18 is least, compared with C(1)=36 and C(6)=26.

6. A garden against a wall has two perpendicular sides costing 2 dollars per metre and its remaining parallel side costing 3 dollars per metre. Budget is 120 dollars. Let the perpendicular side be x and parallel side y. [7 marks]

(a) Form the budget constraint and area function. [3 marks]

Both perpendicular sides cost 4x in total.

Solve for y=40−4x/3, then use A=xy.

Answer: $4x + 3y = 120$; $A = 40x - \frac{4}{3}x^2$, $0 < x < 30$.

(b) Maximise the area. [4 marks]

$A' = 40 - \frac{8}{3}x = 0$ gives 15.

The derivative changes + to −; y=20 and area=300.

Answer: 300m² at x=15m, y=20m.

7. A particle follows $s(t) = t^3 - 6t^2 + 9t$ metres on $0 \le t \le 4$. [6 marks]

(a) When does it move in the negative direction, and what is its speed at t=2? [3 marks]

$v = 3(t-1)(t-3)$ is negative between roots.

v(2)=−3, so speed=3.

Answer: For 1<t<3; speed at 2 is 3m/s.

(b) Find its total distance. [3 marks]

Positions at 0,1,3,4 are 0,4,0,4; sum the absolute changes: 4+4+4.

Answer: 12m.

8. $f(x) = x^3 - ax + b$ touches the x-axis with a horizontal tangent at x=2. [6 marks]

(a) Find a and b. [3 marks]

$f'(2) = 12 - a = 0$ gives a=12.

$f(2) = 8 - 24 + b = 0$ gives b=16.

Answer: a=12, b=16.

(b) Find the other stationary point and classify both. [3 marks]

$f' = 3(x-2)(x+2)$ has signs +,−,+.

f(−2)=−8+24+16=32.

Answer: Maximum (−2,32); minimum (2,0).

9. Positive numbers a and b satisfy a+b=18. Maximise the product ab^2. [6 marks]

(a) Form an objective and state its domain. [2 marks]

Use b=18−a and require both numbers positive.

Answer: $P(a) = a(18-a)^2$, $0 < a < 18$.

(b) Find the greatest product and the corresponding numbers. [4 marks]

$P' = (18-a)(18-3a)$, so the interior stationary input is 6.

The derivative changes + to −; the boundary limits are 0. $P(6)=6\times12^2=864$.

Answer: 864 at a=6, b=12.

10. Consider $M(x) = -x^3 + 12x^2 - 36x + 50$ on $[0, 8]$. [6 marks]

(a) Find all stationary points and classify them. [3 marks]

$M' = -3(x-2)(x-6)$ has signs −,+,−.

M(2)=18 and M(6)=50.

Answer: Minimum (2,18); maximum (6,50).

(b) Find every input giving the global maximum and explain why a local maximum alone is insufficient. [3 marks]

Endpoint M(0)=50 equals the interior maximum; M(8)=18.

A closed-domain comparison must include boundaries, and an optimum need not be unique.

Answer: Global maximum 50 at x=0 and x=6.