Differentiation practice test 5: interpretation and decisions

10 questions | 64 marks | 75 minutes

Original chapter assessment with linked reasoning and applications. Show working, state relevant domains and retain exact values until a numerical answer is requested.

Equivalent correct methods earn the corresponding marks.

1. The table gives values and derivatives of two functions at x=2. [5 marks]

x	u(x)	u prime(x)	v(x)	v prime(x)
2	3	-1	5	4

(a) Find $(uv)'(2)$. [2 marks]

Use $u'v + uv' = (-1)(5) + (3)(4)$.

Answer: 7

(b) Find $(u/v)'(2)$. [3 marks]

Use $[(-1)(5) - (3)(4)]/5^2$.

Answer: $-17/25$

2. Let $f(x) = \sqrt{9 - x^2}$. [6 marks]

(a) State the real function domain and find its derivative. [3 marks]

The radicand must be nonnegative.

Use the chain rule and note where the derivative denominator is zero.

Answer: $-3 \le x \le 3$; $f' = -x/\sqrt{9 - x^2}$ for $-3 < x < 3$

(b) Find the tangent at zero and explain why the endpoints are not stationary points. [3 marks]

The derivative is zero at 0 and the height is 3.

Stationary means a defined zero derivative, not merely an endpoint.

Answer: $y = 3$; the endpoint derivatives are not finite.

3. Let $g(x) = x^2 e^{-x}$ for real x. In part (b), restrict to $0 \le x \le 4$. [7 marks]

(a) Find and classify the stationary points. [4 marks]

$g' = xe^{-x}(2 - x)$.

The derivative changes negative-to-positive at 0 and positive-to-negative at 2.

Answer: Local minimum $(0, 0)$; local maximum $(2, 4e^{-2})$.

(b) Find the absolute maximum on the interval. [3 marks]

Compare heights at 0, 2 and 4.

The endpoint value $16e^{-4}$ is smaller than $4e^{-2}$.

Answer: $4e^{-2}$ at $x = 2$

4. For $a > 0$, the identity $a^x = e^{x \ln a}$ is supplied. [5 marks]

(a) Derive a formula for the derivative of a^x. [3 marks]

Apply the chain rule to $e^{x \ln a}$.

The exponent derivative is the constant $\ln a$.

Answer: $(\ln a) a^x$

(b) Find the tangent to $y = 3^x$ at zero. [2 marks]

At zero the height is 1 and the gradient is $\ln 3$.

Answer: $y = (\ln 3)x + 1$

5. Analyse $f(x) = x^5 - 5x$. [6 marks]

(a) Find and classify the stationary points. [4 marks]

$f' = 5(x^4 - 1)$ has real roots ± 1.

The derivative changes positive-to-negative at -1 and negative-to-positive at 1.

Answer: Maximum $(-1, 4)$; minimum $(1, -4)$.

(b) Find and justify its inflection point. [2 marks]

$f'' = 20x^3$ changes sign at zero.

Answer: $(0, 0)$

6. Let $h(x) = 1/(1 + x^2)$. [7 marks]

(a) Find its first derivative. [2 marks]

Use a negative power and the chain rule.

Answer: $-2x/(1 + x^2)^2$

(b) Show that $h'' = (6x^2 - 2)/(1 + x^2)^3$, then find the inflection points. [5 marks]

Differentiate $-2x(1 + x^2)^{-2}$ with product and chain rules.

The positive denominator means the numerator controls concavity.

It changes sign at both roots of $6x^2 - 2$.

Answer: $(-1/\sqrt{3}, 3/4)$ and $(1/\sqrt{3}, 3/4)$

7. A height is $H(t) = 10 + 4t - t^2$ metres for $0 \le t \le 4$ seconds. [6 marks]

(a) Find its average rate from 0 to 2 seconds. [2 marks]

Use $(H(2) - H(0))/2 = (14 - 10)/2$.

Answer: 2 m/s

(b) Find when its instantaneous rate equals that average. [2 marks]

$H' = 4 - 2t$; set it equal to 2.

Answer: $t = 1$ s

(c) Find its maximum height. [2 marks]

The derivative is zero at 2 and the quadratic is concave down.

Answer: 14 m at 2 s.

8. A rectangular enclosure is divided into three equal pens by two internal fences parallel to its width W. There is 120 m of fencing for the boundary and the two dividers. [6 marks]

(a) Express length L and total area in terms of W. [3 marks]

The total fence length is $2L + 4W = 120$.

Answer: $L = 60 - 2W$; $A = 60W - 2W^2$, $0 < W < 30$

(b) Find dimensions giving greatest total area. [3 marks]

Set $A' = 60 - 4W = 0$.

The second derivative is negative.

Answer: L=30 m and W=15 m; total area 450 square metres.

9. An open cylindrical container has volume 36π cubic centimetres. It has a base but no lid. [7 marks]

(a) Write its surface area as a function of radius r. [3 marks]

The volume constraint gives $h = 36/r^2$.

Use base plus curved surface area.

Answer: $S = \pi r^2 + 72\pi/r$, $r > 0$

(b) Find the dimensions minimising material area. [4 marks]

Set $S' = 2\pi r - 72\pi/r^2 = 0$, obtaining $r^3 = 36$.

Substitute for h; the area grows without bound near zero and for large radius.

Answer: $r = h = \sqrt[3]{36}$ cm

10. A damped displacement model is $s(t) = e^{-t}\cos(2t)$ metres, $t \geq 0$. [9 marks]

(a) Find velocity and acceleration. [4 marks]

Use product and chain rules twice.

Answer: $v = e^{-t}[-\cos(2t) - 2\sin(2t)]$; $a = e^{-t}[-3\cos(2t) + 4\sin(2t)]$

(b) Is the particle speeding up at zero? [2 marks]

At zero, velocity is -1 m/s and acceleration is -3 m/s squared.

The signs agree.

Answer: Yes.

(c) Find its first positive rest time to 3 significant figures. [3 marks]

Solve $\tan(2t) = -1/2$.

Choose the first positive angle in quadrant II.

Answer: $[\pi - \arctan(1/2)]/2 \approx 1.34$ s