Differentiation practice test 6: synthesis and applications

10 questions | 76 marks | 75 minutes

Original chapter assessment with linked reasoning and applications. Show working, state relevant domains and retain exact values until a numerical answer is requested.

Equivalent correct methods earn the corresponding marks.

1. Differentiate efficiently, stating any excluded inputs. [5 marks]

(a) $f(x) = (x^2 - 1)/(x + 1)$. [2 marks]

Simplify to $x - 1$ only where the original denominator is nonzero.

Answer: $f' = 1$ for $x \neq -1$

(b) $g(x) = (x^2 + 1)^3 e^{-x}$. [3 marks]

Use product and chain rules.

Factor the common exponential and power.

Answer: $g' = e^{-x}(x^2 + 1)^2(6x - x^2 - 1)$

2. A temperature model is $T(t) = 18 + 6\sin[\pi(t-4)/12]$ degrees Celsius, for $0 \le t \le 24$ hours. [9 marks]

(a) Find the maximum and minimum temperatures and their times. [3 marks]

The sine is 1 at angle $\pi/2$ and -1 at $3\pi/2$.

Solve for t within the stated interval.

Answer: Maximum 24 degrees Celsius at t=10; minimum 12 degrees Celsius at t=22.

(b) Find when warming is fastest, and the rate. [3 marks]

$T' = (\pi/2)\cos[\pi(t-4)/12]$.

Its maximum occurs when the cosine is 1.

Answer: $t = 4$ h; $\pi/2$ degrees Celsius per hour

(c) Find both times when the temperature is 20 degrees Celsius. [3 marks]

Solve sine equal to 1/3.

Use the two permitted angles in the displayed period.

Answer: $4 + (12/\pi)\arcsin(1/3)$ and $16 - (12/\pi)\arcsin(1/3)$ hours

3. Let $f(x) = x^3 - 3ax$, with $a > 0$. [7 marks]

(a) Find stationary coordinates in terms of a. [3 marks]

$f' = 3(x^2 - a)$; substitute $x = \pm\sqrt{a}$ into f.

Answer: $(-\sqrt{a}, 2a^{3/2})$ and $(\sqrt{a}, -2a^{3/2})$

(b) Classify them. [2 marks]

$f'' = 6x$ is respectively negative and positive.

Answer: The negative-input point is a maximum; the positive-input point is a minimum.

(c) Their vertical separation is 32. Find a. [2 marks]

$4a^{3/2} = 32$ gives $a^{3/2} = 8$.

Answer: $a = 4$

4. A cooling object follows $T(t) = 20 + 80e^{-kt}$ degrees Celsius, with t in minutes. After 5 minutes its temperature is 60 degrees Celsius. [8 marks]

(a) Find k exactly. [3 marks]

$40 = 80e^{-5k}$, so $e^{-5k} = 1/2$.

Answer: $k = \ln 2/5$ per minute

(b) Find the initial rate of temperature change. [3 marks]

$T' = -80ke^{-kt}$; substitute zero and the value of k.

Answer: $-16\ln 2 \approx -11.1$ degrees Celsius per minute

(c) Find when its temperature reaches 30 degrees Celsius. [2 marks]

$10 = 80e^{-kt}$ gives $kt = \ln 8 = 3\ln 2$.

Answer: 15 minutes.

5. The curve $y = (x^2 + ax + b)/(x - 1)$ has oblique asymptote $y = x + 3$ and y-intercept -4. [9 marks]

(a) Find a and b. [3 marks]

Polynomial division gives oblique asymptote $y = x + a + 1$.

At zero the function value is $-b$.

Answer: $a = 2$, $b = 4$.

(b) Find stationary coordinates. [4 marks]

The function is $x + 3 + 7/(x - 1)$.

The derivative $1 - 7/(x - 1)^2$ is zero when $(x - 1)^2 = 7$.

Answer: $(1 - \sqrt{7}, 4 - 2\sqrt{7})$ and $(1 + \sqrt{7}, 4 + 2\sqrt{7})$

(c) Classify them. [2 marks]

The second derivative is $14/(x - 1)^3$, negative left of 1 and positive right.

Answer: The left point is a local maximum; the right is a local minimum.

6. A closed cylindrical tank has volume 200π cubic metres. Material for its top and base costs twice as much per square metre as material for its curved side. [7 marks]

(a) Give a cost function proportional to the actual cost, in terms of radius r. [3 marks]

Use $h = 200/r^2$.

Weight the combined top/base area by 2 and add side area.

Answer: $C(r) = 4\pi r^2 + 400\pi/r$, $r > 0$

(b) Find the optimum relation between height and radius. [4 marks]

The derivative equation is $8\pi r - 400\pi/r^2 = 0$.

Thus $r^3 = 50$ and $h = 200/r^2 = 4r$.

The cost tends to infinity at both extremes, so the stationary minimum is absolute.

Answer: $r = \sqrt[3]{50}$ m and $h = 4r$

7. A particle has $s(t) = t^4/4 - 2t^3 + 9t^2/2 - 4t$ metres, $0 \le t \le 5$ seconds. [9 marks]

(a) Factorise the velocity and find its rest times. [3 marks]

Differentiate position and factor the resulting cubic.

Answer: $v = (t-1)^2(t-4)$; rest at 1 s and 4 s.

(b) Which rest time is a reversal? [2 marks]

The squared factor at 1 does not change sign.

The velocity changes negative-to-positive at 4.

Answer: Only t=4 s.

(c) Find displacement and total distance. [4 marks]

Positions at $0, 4, 5$ are $0, -8, -5/4$.

The unchanged direction at 1 needs no extra distance segment.

Distance is $8 + 27/4 = 59/4$.

Answer: Displacement $-5/4$ m; total distance $59/4$ m

8. A rectangle in the first quadrant has its upper-right corner on $y = 12e^{-x}$ and its lower-left corner at the origin. Its sides are parallel to the coordinate axes. [6 marks]

(a) Form the area function and find its derivative. [3 marks]

Area is width times height.

Use the product rule.

Answer: $A = 12xe^{-x}$; $A' = 12e^{-x}(1 - x)$

(b) Find and justify its maximum area. [3 marks]

The derivative changes positive-to-negative at 1.

The area tends to zero as x tends to zero or infinity.

Answer: $12/e$ square units at width 1

9. A smooth function satisfies $f'(x) = (x^2 - 1)e^{-x}$. [8 marks]

(a) Find and classify the stationary inputs. [3 marks]

The exponential is positive and $x^2 - 1$ changes signs positive, negative, positive.

Answer: Maximum at x=-1; minimum at x=1.

(b) Find the inputs where concavity changes. [3 marks]

$f'' = e^{-x}(-x^2 + 2x + 1)$.

The quadratic changes sign at both roots.

Answer: $1 - \sqrt{2}$ and $1 + \sqrt{2}$

(c) Can the stationary heights be determined from the supplied information alone? [2 marks]

An arbitrary added constant leaves the derivative unchanged.

Answer: No.

10. A product has demand price $p(q) = 80 - 0.5q$ dollars per unit and cost $C(q) = 20q + 200$ dollars, where $0 \leq q \leq 160$. [8 marks]

(a) Form profit as a function of output. [2 marks]

Revenue is $qp(q)$; subtract the cost.

Answer: $\Pi(q) = 60q - 0.5q^2 - 200$

(b) Find the profit-maximising output, price and profit. [4 marks]

The derivative is $60 - q$, zero at 60.

The quadratic is concave down.

Substitute into price and profit.

Answer: 60 units; price 50 dollars per unit; profit 1600 dollars.

(c) A student proposes 80 units by maximising revenue. Explain why that does not maximise profit. [2 marks]

Revenue is maximised at 80, where its marginal rate is zero.

Profit subtracts a cost increasing at 20 dollars per unit.

Answer: The marginal cost is positive, so the revenue maximum ignores extra production costs.