Logarithmic functions - Practice test 4

10 questions | 54 marks | 70 minutes

Reasoning about domains, parameters, inverse graphs and an optimisation model.

Equivalent correct methods earn the corresponding marks.

1. Find real domains before simplifying. [5 marks]

(a) State the domain of $\ln(x+2)+\ln(4-x)$. [2 marks]

Both logarithm inputs must be positive.

Answer: $-2<x<4$.

(b) Compare it with the domain of $\ln((x+2)(4-x))$. [3 marks]

The product is positive between its zeros -2 and 4 and negative outside.

Answer: The same domain, $-2<x<4$, in this particular case.

2. Determine unknown logarithm bases. [5 marks]

(a) Find b if $\log_b 9 = 1/2$. [2 marks]

$b^{1/2}=9$, so square both sides.

Answer: $b=81$

(b) Find c if $\log_c 16 = -2$. [3 marks]

$c^{-2}=16$, giving positive $c=1/4$.

Answer: $c=1/4$

3. For $u > 0$, define $H(u) = \ln u + \ln(1/u)$. [5 marks]

(a) Show that H(u) is constant and state its value. [2 marks]

The product of the inputs is 1, so $H = \ln 1 = 0$.

Answer: $H(u) = 0$.

(b) Differentiate the original two-term expression to confirm the result. [3 marks]

For the second term, divide $-1/u^2$ by $1/u$.

Answer: $H'(u) = 1/u - 1/u = 0$.

4. Solve $\ln(x^2 - 3x) = \ln 4$. [6 marks]

(a) Find the original domain. [3 marks]

$x(x - 3) > 0$.

Answer: $x < 0$ or $x > 3$.

(b) Find and check every real solution. [3 marks]

$x^2 - 3x - 4 = (x - 4)(x + 1) = 0$; both candidates make the input equal to 4.

Answer: $x = -1$ or 4.

5. Consider $2^x + 2^{-x} = 5/2$. [6 marks]

(a) Use u = 2^x to find a quadratic equation in u. [3 marks]

$u + 1/u = 5/2$; multiply by $2u$.

Answer: $2u^2 - 5u + 2 = 0$, with $u > 0$.

(b) Find all real x. [3 marks]

$(2u - 1)(u - 2) = 0$ gives $u = 1/2, 2$.

Answer: $x = 1$ or -1.

6. Let $f(x) = \ln(3x)$, $x > 0$. [5 marks]

(a) Find its inverse and describe the relationship between the two graphs. [3 marks]

$y = \ln(3x)$ implies $x = e^y/3$.

Answer: $f^{-1}(x) = e^x/3$; reflection in $y = x$.

(b) Find the point on the inverse corresponding to (1/3, 0) on f. [2 marks]

An inverse interchanges input and output.

Answer: $(0, 1/3)$.

7. Let $f(x) = a\ln x + bx$ for $x > 0$. Its tangent at x = 1 is horizontal and f(1) = 6. [6 marks]

(a) Determine a and b. [3 marks]

$f(1) = b = 6$. The derivative is $a/x + b$, so $a + b = 0$.

Answer: $a = -6$, $b = 6$.

(b) Classify the stationary point at x = 1. [3 marks]

$f'' = -a/x^2 = 6/x^2 > 0$.

Answer: A local minimum at (1,6).

8. A mathematical cost model is $C(x) = 20 + x - 4\ln x$ for $x > 0$. [6 marks]

(a) Find the value of x minimising C and justify that it is a global minimum. [3 marks]

$C' = 1 - 4/x$ is negative on $(0, 4)$ and positive on $(4, \infty)$.

Answer: $x = 4$.

(b) Give the minimum exactly and explain why x = 0 cannot be tested as an endpoint. [3 marks]

The domain is open at zero; the function tends to infinity as $x \to 0^+$.

Answer: $24 - 4\ln 4$; zero is outside the logarithm domain.

9. Distance travelled in a supplied model is $D(t) = 12\ln(t+1)$ metres, $t \geq 0$ seconds. [5 marks]

(a) Find the time when D = 12 metres. [2 marks]

$\ln(t+1) = 1$.

Answer: $t = e - 1$ seconds.

(b) If t + 1 doubles, show that the additional distance is constant and find it. [3 marks]

$12\ln(2(t+1)) - 12\ln(t+1) = 12\ln 2$.

Answer: $12\ln 2$ metres.

10. A student fits a formula without checking its useful domain. [5 marks]

(a) Find the mathematical domain of $M(t) = 50 + 10\ln(t-2)$. [2 marks]

Require $t - 2 > 0$.

Answer: $t > 2$.

(b) The data were observed only for $3 \leq t \leq 8$ days. Explain two separate cautions about predicting at t = 1 and t = 100. [3 marks]

Distinguish a mathematical domain restriction from limited empirical validity.

Answer: At t = 1 the formula is not real; at t = 100 it is defined but is a large extrapolation.