Logarithmic functions - Practice test 6

10 questions | 61 marks | 70 minutes

A more demanding paper combining precise domains, parameter choices and mathematical reasoning.

Equivalent correct methods earn the corresponding marks.

1. Solve and justify the base restrictions. [5 marks]

(a) Find b if $\log_b(1/16) = 4$. [2 marks]

$b^4 = 1/16$ with $b > 0$; the positive fourth root is 1/2.

Answer: $b = 1/2$.

(b) Explain why base 1 cannot define a logarithm on all positive inputs. [3 marks]

An inverse function requires a unique output for every allowed input.

Answer: $1^y = 1$ for every y, so inputs other than 1 are never produced and input 1 has no unique exponent.

2. Consider $F(x) = \ln(x - 1) + \ln(x + 3)$. [6 marks]

(a) State its domain and solve F(x) = ln(12). [4 marks]

$(x - 1)(x + 3) = 12$ gives $x^2 + 2x - 15 = 0$; its roots are $3, -5$. Only 3 is valid.

Answer: Domain $x > 1$; solution $x = 3$.

(b) Explain why solving the combined equation may introduce an inadmissible branch. [2 marks]

Original-domain checking removes the negative branch.

Answer: The combined positive-product condition also allows $x < -3$, whereas both original inputs must be positive.

3. Solve $e^x + 3e^{-x} = 4$. [5 marks]

(a) Find all possible positive values of u = e^x. [3 marks]

$u + 3/u = 4$ gives $u^2 - 4u + 3 = 0$.

Answer: $u = 1$ or 3.

(b) Find all x and decide which is larger. [2 marks]

Apply the natural logarithm to each positive u.

Answer: $x = 0$ and $x = \ln 3$; $\ln 3 > 0$.

4. A family is $f_a(x) = \ln(x - a) + 2a$. [6 marks]

(a) Find the domain and x-intercept in terms of a. [3 marks]

Solve $\ln(x - a) = -2a$.

Answer: $x > a$; intercept $(a + e^{-2a}, 0)$.

(b) Find the inverse function and state its range. [3 marks]

Solve $y - 2a = \ln(x - a)$ for x and interchange names.

Answer: $f_a^{-1}(x) = a + e^{x - 2a}$; range (a, ∞).

5. Let $g(x) = \ln x - kx$ for x > 0 and k > 0. [7 marks]

(a) Find the stationary point in terms of k and classify it. [4 marks]

$g' = 1/x - k$ and $g'' = -1/x^2 < 0$.

Answer: $x = 1/k$, $y = -\ln k - 1$; a maximum.

(b) Find k so that the maximum value is zero. [3 marks]

$-\ln k - 1 = 0$ implies $\ln k = -1$.

Answer: $k = e^{-1}$.

6. Consider $h(x) = \ln((x+1)/(x-1))$. [6 marks]

(a) Find its real domain. [3 marks]

The ratio must be positive, and its denominator must be nonzero; a sign chart gives the two intervals.

Answer: $x < -1$ or $x > 1$.

(b) Differentiate h on its entire domain without splitting it into invalid logarithms. [3 marks]

For $u = (x+1)/(x-1)$, $u' = -2/(x-1)^2$. Then $u'/u = -2/((x-1)(x+1))$.

Answer: $h'(x) = -2/(x^2-1)$.

7. The curve is $y = \ln(2 + x^2)$. [7 marks]

(a) Find the tangent at x = 1. [3 marks]

The derivative is $2x/(2+x^2)$, so the gradient is 2/3.

Answer: $y - \ln 3 = \frac{2}{3}(x-1)$.

(b) Find the two points at which the tangent gradient equals 1/2. [4 marks]

Solve $2x/(2+x^2) = 1/2$, giving $x^2 - 4x + 2 = 0$. Use $2 + x^2 = 4x$ at these roots.

Answer: $x = 2 \pm \sqrt{2}$; the corresponding y-values are $\ln(8 \pm 4\sqrt{2})$.

8. A response is $R(t) = A + B\ln(1 + t/c)$ with B,c > 0 and t ≥ 0. Time is in hours. [7 marks]

(a) Show that its derivative is B/(c+t) and give the rate units. [3 marks]

Apply the chain rule: $B(1/c)/(1 + t/c)$.

Answer: $R'(t) = B/(c+t)$, response units per hour.

(b) If the initial rate is 6 and the rate at t = 4 is 2, find B and c. [4 marks]

$B/c = 6$ gives $B = 6c$. Then $B/(c+4) = 2$ gives $6c = 2c + 8$.

Answer: $c = 2$, $B = 12$.

9. A model score is $S(x) = 10\log_{10}(x/x_0)$ for positive x and x0. [6 marks]

(a) A change of 7 scale units occurs. Find the exact multiplier of x. [3 marks]

$7 = 10\log_{10}(x_2/x_1)$, so divide by 10 and exponentiate.

Answer: $10^{0.7}$.

(b) Show that adding equal amounts to x does not generally give equal changes in S. [3 marks]

Subtract the scales at $x + d$ and x, for positive inputs.

Answer: The change for adding d is $10\log_{10}(1 + d/x)$, which depends on x.

10. A proposed simplification and an asymptote need careful interpretation. [6 marks]

(a) Compare the functions $p(x) = \ln(x^2 - 1)$ and $q(x) = \ln(x - 1) + \ln(x + 1)$. Where do they agree and where is only p defined? [3 marks]

p needs a positive product, while q needs both factors positive.

Answer: They agree for $x > 1$; only p is defined for $x < -1$.

(b) For p, find its derivative and the limiting behaviour as $x \to -1^-$ and $x \to 1^+$. [3 marks]

The positive input tends to zero at each boundary, so its natural logarithm tends to negative infinity.

Answer: $p' = 2x/(x^2 - 1)$; both limits are $-\infty$.