Discrete distributions - Practice test 2

10 questions | 64 marks | 70 minutes

Compare models, recover a distribution from cumulative information and examine the role of independence.

Equivalent correct methods earn the corresponding marks.

1. A bag contains two red and one blue counter. Two counters are selected fairly without replacement. X is the red count. [8 marks]

(a) Construct the distribution of X. [3 marks]

The three equally likely unordered pairs contain either both red counters or one red and the blue.

Two of the three pairs contain exactly one red.

Answer: $P(X = 0) = 0,\ P(X = 1) = 2/3,\ P(X = 2) = 1/3.$

(b) Calculate E(X). [2 marks]

$E(X) = 1(2/3) + 2(1/3) = 4/3.$

Answer: 4/3.

(c) Explain why a Bin(2, 2/3) model has the correct mean but is not the correct distribution. [3 marks]

Bin(2, 2/3) predicts P(X = 2) = 4/9 rather than 1/3.

Answer: The mean matches, but replacement is absent and trials are dependent.

2. X takes only −2, 0 and 5. Its cumulative probabilities at these values are 0.2, 0.7 and 1. [6 marks]

(a) Find P(X = 0). [2 marks]

Subtract the previous cumulative value: 0.7 − 0.2.

Answer: 0.5.

(b) Find P(−1 < X < 5). [2 marks]

Only X = 0 lies strictly between −1 and 5.

Answer: 0.5.

(c) Find F(3). [2 marks]

No probability mass occurs between 0 and 5, so the cumulative value stays 0.7.

Answer: 0.7.

3. X takes values 0, 2 and 8 with probabilities a, 0.5 and 0.5 − a. Its expected value is 3. [6 marks]

(a) Find a. [3 marks]

E(X) = 2(0.5) + 8(0.5 − a) = 5 − 8a.

Solve 5 − 8a = 3.

Answer: a = 0.25.

(b) Find Var(X). [2 marks]

$E(X^2) = 4(0.5) + 64(0.25) = 18$.

$Var(X) = 18 - 3^2 = 9$.

Answer: 9.

(c) Check that all probabilities are valid. [1 mark]

Each is non-negative and their sum is 1.

Answer: They are 0.25, 0.5, 0.25.

4. Option A pays 0 or 2 with equal probabilities. Option B always pays 1. [6 marks]

(a) Show that their means are equal. [2 marks]

For A, $E(A) = 0(0.5) + 2(0.5) = 1$; B is constant at 1.

Answer: Both means are 1.

(b) Compare their variances. [2 marks]

$E(A^2) = 2$, giving variance $2 - 1 = 1$.

A constant has zero variance.

Answer: Var(A) = 1; Var(B) = 0.

(c) Which option avoids the possibility of receiving zero? Explain. [2 marks]

A pays zero with probability 0.5; B always pays 1.

Answer: Option B.

5. X takes 0, 1 and 3 with probabilities 0.1, 0.2 and 0.7. Let $Y = 10 - 2X$. [7 marks]

(a) Give the values and probabilities of Y. [2 marks]

Apply $10 - 2x$ to each value and retain its probability.

Answer: Y = 10, 8, 4 with probabilities 0.1, 0.2, 0.7.

(b) Find $P(Y \geq 8)$. [2 marks]

Use $Y = 10$ or $Y = 8$, hence $0.1 + 0.2$.

Answer: 0.3.

(c) Find the mean and variance of Y. [3 marks]

$E(X) = 2.3$; $E(X^2) = 6.5$; $Var(X) = 6.5 - 2.3^2 = 1.21$.

$E(Y) = 10 - 2(2.3) = 5.4$; $Var(Y) = 4(1.21)$.

Answer: Mean 5.4; variance 4.84.

6. Independent success indicators B and C have success probabilities 0.2 and 0.6. Set S = B + C. [7 marks]

(a) Find the distribution of S. [3 marks]

No success: 0.8(0.4). Both: 0.2(0.6).

One success: 0.2(0.4) + 0.8(0.6).

Answer: P(S = 0) = 0.32; P(S = 1) = 0.56; P(S = 2) = 0.12.

(b) Find its mean and variance. [2 marks]

Add Bernoulli means: 0.2 + 0.6.

Add independent variances: 0.16 + 0.24.

Answer: Mean 0.8; variance 0.4.

(c) Why is S not Bin(2, 0.4)? [2 marks]

Bin(2, 0.4) would have variance 2(0.4)(0.6) = 0.48, not 0.4.

Answer: The success probabilities are different.

7. A fair coin is tossed once. B = 1 for heads and C is defined to equal B. Let S = B + C. [6 marks]

(a) Construct the distribution of S and find Var(S). [2 marks]

E(S) = 1 and E(S^2) = 2, so Var(S) = 1.

Answer: S is 0 or 2 with equal probabilities; variance 1.

(b) Why is adding the two Bernoulli variances invalid here? [2 marks]

They are always equal; their sum variance is not 0.25 + 0.25 = 0.5.

Answer: B and C are dependent.

(c) Does E(S) = E(B) + E(C) still hold? Explain. [2 marks]

Additivity of expectation does not require independence.

Answer: Yes; both sides equal 1.

8. A group has 3 marked and 17 unmarked items. Two are sampled without replacement. [6 marks]

(a) Find the exact probability that both are marked. [2 marks]

$(3/20)(2/19) = 3/190$.

Answer: $3/190 \approx 0.0158$.

(b) Find the probability predicted by Bin(2, 0.15) for two marked items. [2 marks]

The independent model gives 0.15^2.

Answer: 0.0225.

(c) Explain the source of the difference. [2 marks]

Given a marked first item, the next marked probability is 2/19, not 3/20.

Answer: The second draw changes the available composition.

9. $X \sim \text{Bin}(6, 0.4)$. Give probabilities to four decimal places. [6 marks]

(a) Find P(1 ≤ X ≤ 3). [2 marks]

$\sum_{k=1}^{3} \binom{6}{k}(0.4)^k(0.6)^{6-k} = 0.7741.$

Answer: 0.7741

(b) Find P(X > 3). [2 marks]

Use 1 − P(X ≤ 3).

$\sum_{k=4}^{6} \binom{6}{k}(0.4)^k(0.6)^{6-k} = 0.1792.$

Answer: 0.1792

(c) Among 80 comparable groups, how many are expected to have more than three successes? [2 marks]

Multiply the unrounded probability in (b) by 80.

Answer: 14.3360

10. A device works if at least 3 of its 4 independent components work. Each component works with probability p. [6 marks]

(a) Find the device reliability for p = 0.9. [2 marks]

Let X be Bin(4, 0.9).

$P(X \geq 3) = 4(0.9^3)(0.1) + 0.9^4$.

Answer: 0.9477.

(b) Find the reliability for p = 0.95. [2 marks]

Use Bin(4, 0.95): $4(0.95^3)(0.05) + 0.95^4$.

Answer: 0.98598125.

(c) Which component choice meets a device target of at least 0.98? [2 marks]

0.9477 < 0.98 whereas 0.98598125 ≥ 0.98.

Answer: The p = 0.95 components.