Discrete distributions - Practice test 4

10 questions | 64 marks | 70 minutes

Explain distribution identities, distinguish dependent indicators and solve a minimum-trial problem.

Equivalent correct methods earn the corresponding marks.

1. X takes values 0, 1, 2 and 3, with $P(X = x) = k(x + 1)$. [6 marks]

(a) Find k and P(X ≥ 2). [2 marks]

The weights sum to 1 + 2 + 3 + 4 = 10, so k = 0.1.

The final two weights are 3 and 4.

Answer: k = 0.1; P(X ≥ 2) = 0.7.

(b) Find E(X) and Var(X). [2 marks]

E(X) = 0.2 + 0.6 + 1.2 = 2.

E(X²) = 0.2 + 1.2 + 3.6 = 5; variance = 5 − 4.

Answer: Mean 2; variance 1.

(c) Find F(1.5). [2 marks]

Only 0 and 1 are no larger than 1.5; add 0.1 + 0.2.

Answer: 0.3.

2. X takes values 0, 1, 2 and 3. Its cumulative values are F(0) = 0.15, F(1) = a, F(2) = 0.8 and F(3) = 1. Also P(X = 1) = 0.35. [6 marks]

(a) Find a. [2 marks]

The jump at 1 is a − 0.15 = 0.35.

Answer: a = 0.5.

(b) Find P(X = 2). [2 marks]

F(2) − F(1) = 0.8 − 0.5.

Answer: 0.3.

(c) Find P(X ≥ 2) and explain the endpoint. [2 marks]

Use 1 − F(1), since the value 2 must remain included.

Answer: 0.5.

3. X takes −2, 0 and 2 with probabilities 1/4, 1/2 and 1/4. Let Y = X^2. [7 marks]

(a) Find E(X), E(X^2) and Var(X). [2 marks]

The symmetric distribution has mean 0.

E(X^2) = 4(1/4) + 4(1/4) = 2; variance is 2 − 0.

Answer: 0, 2 and 2.

(b) Construct the distribution of Y and find its variance. [3 marks]

Combine X = −2 and X = 2 at the same transformed value 4.

E(Y) = 2; E(Y^2) = 16(1/2) = 8; variance = 8 − 4.

Answer: P(Y = 0) = 1/2, P(Y = 4) = 1/2; Var(Y) = 4.

(c) Explain why E(X^2) is not $[E(X)]^2$ here. [2 marks]

Expectation does not generally commute with a nonlinear operation.

Answer: They are 2 and 0, respectively.

4. D is the score on a fair six-sided die. [6 marks]

(a) Calculate E(D²). [2 marks]

$E(D^2) = (1 + 4 + 9 + 16 + 25 + 36)/6 = 91/6$.

Answer: 91/6.

(b) Use E(D) = 3.5 to find Var(D). [2 marks]

$91/6 - (7/2)^2 = 182/12 - 147/12 = 35/12$.

Answer: 35/12.

(c) For two independent dice, find the variance of their total. [2 marks]

Add the two independent variances: 2(35/12).

Answer: 35/6.

5. X has mean $\mu = 30$ and standard deviation $\sigma = 6$. Define $Z = (X - 30)/6$. [6 marks]

(a) Find E(Z) and Var(Z). [2 marks]

Mean = (30 − 30)/6 = 0.

Variance = 36/6² = 1.

Answer: 0 and 1.

(b) Express the event Z > 2 as an event involving X. [2 marks]

Multiply by positive 6 and add 30: X − 30 > 12.

Answer: X > 42.

(c) Can P(Z > 2) be obtained from the supplied mean and SD alone? [2 marks]

The full distribution has not been specified; standardising does not automatically make it normal.

Answer: No.

6. Three independent indicators have success probabilities 0.1, 0.4 and 0.8. T is their total. [6 marks]

(a) Find its mean and variance. [2 marks]

Add the three means.

Add p(1 − p): 0.09 + 0.24 + 0.16 = 0.49.

Answer: Mean 1.3; variance 0.49.

(b) Find P(T = 0) and P(T = 3). [2 marks]

No success: 0.9(0.6)(0.2).

All succeed: 0.1(0.4)(0.8).

Answer: 0.108 and 0.032.

(c) Explain why matching the mean with Bin(3, 1.3/3) does not establish that model. [2 marks]

For example, the binomial variance would be 1.3(1 − 1.3/3), not 0.49.

Answer: The trial probabilities are not equal.

7. A fair coin defines B = 1 for heads. Let C = 1 − B and T = B + C. [6 marks]

(a) Find E(T) and Var(T) directly. [2 marks]

T is identically 1 whatever the coin outcome.

Answer: Mean 1; variance 0.

(b) Calculate what an incorrect independence assumption would give for Var(T). [2 marks]

Each individual indicator has variance 1/4.

Adding them as if independent would give 1/2.

Answer: 0.5.

(c) Explain the dependence. [2 marks]

If B = 1 then C = 0, and conversely.

Answer: Knowing B fixes C exactly.

8. $X \sim \text{Bin}(2, p)$, where $0 < p < 1$, and $P(X = 2) = 9P(X = 0)$. [7 marks]

(a) Form and solve an equation for p. [3 marks]

$p^2 = 9(1 - p)^2$.

Both p and $1 - p$ are positive, so $p = 3(1 - p)$, giving $p = 3/4$.

Answer: $p = 3/4$.

(b) Find P(X = 1). [2 marks]

$2p(1 - p) = 2(3/4)(1/4)$.

Answer: 3/8.

(c) Find the mean and variance. [2 marks]

$np = 2(3/4)$; $np(1 - p) = 2(3/4)(1/4)$.

Answer: Mean 3/2; variance 3/8.

9. Independent trials have success probability 0.2. A planner wants the probability of at least one success to be at least 0.95. [7 marks]

(a) Write the required inequality in n. [2 marks]

Use the complement of zero successes.

Answer: $1 - 0.8^n \geq 0.95$.

(b) Find the smallest integer n. [3 marks]

The inequality is $0.8^n \leq 0.05$.

$n \geq \ln(0.05)/\ln(0.8) \approx 13.425$; round up.

Answer: 14.

(c) Verify the boundary using $n - 1$ and n. [2 marks]

The first probability is below 0.95 and the second is above it.

Answer: n = 13 gives 0.9450; n = 14 gives 0.9560.

10. Game A charges 2 dollars and pays 10 dollars with probability 0.2, otherwise 0 dollars. Game B charges 2 dollars and always returns 2 dollars. A player can choose either. [7 marks]

(a) Compare expected net gains. [2 marks]

A has expected prize $10(0.2) = 2$ and fee 2.

B returns exactly its fee.

Answer: Both are zero.

(b) Compare net-gain variances. [2 marks]

For A, gross $E(X^2) = 100(0.2) = 20$ and mean 2, giving variance 16.

Subtracting the fee leaves variance unchanged.

Answer: Game A: 16 dollars^2; Game B: 0.

(c) For ten independent plays of A, find the probability of a strictly positive total net gain. [3 marks]

If K is the number of wins, total net gain is $10K - 20$.

Strictly positive means $K \geq 3$, with $K \sim \text{Bin}(10, 0.2)$.

$$\sum_{k=3}^{10} \binom{10}{k}(0.2)^k(0.8)^{10-k} = 0.3222.$$

Answer: 0.3222