Marking guide | Discrete probability distributions

Discrete distributions - Applications and reasoning test

10 questions | 72 marks | 75 minutes

An integrated paper requiring distribution construction, nonlinear costs, model criticism and justified decisions.

Equivalent correct methods earn the corresponding marks.

1. A count X takes values 0, 1 and 3 with probabilities a, b and 0.2. Its expected value is 1.1. [7 marks]

(a) Find a and b. [3 marks]

Expectation gives $b + 3(0.2) = 1.1$, hence $b = 0.5$.

Total probability gives $a = 1 - 0.5 - 0.2$.

Answer: $a = 0.3$, $b = 0.5$.

(b) Find Var(X). [2 marks]

$E(X^2) = 0.5 + 9(0.2) = 2.3$.

Variance $= 2.3 - 1.1^2$.

Answer: 1.09.

(c) Find $P(X \leq 1)$ and state whether observing 3 is impossible. [2 marks]

Add a + b for the cumulative event.

Answer: 0.8; observing 3 remains possible with probability 0.2.

2. An integer-valued X has F(0) = a, F(1) = 3a, F(2) = 1 − a and F(3) = 1. No other values occur. Also P(X = 1) = 0.4. [7 marks]

(a) Find a and the probability table. [3 marks]

The jump at 1 is 3a − a = 0.4, so a = 0.2.

Successive jumps are 0.2, 0.4, 0.2 and 0.2.

Answer: a = 0.2; probabilities 0.2, 0.4, 0.2, 0.2 at 0, 1, 2, 3.

(b) Find the mean and variance. [2 marks]

E(X) = 0.4 + 0.4 + 0.6 = 1.4.

$E(X^2) = 0.4 + 0.8 + 1.8 = 3$; variance = 3 − 1.96.

Answer: Mean 1.4; variance 1.04.

(c) Find P(1 < X ≤ 3). [2 marks]

Use 1 − F(1) = 1 − 0.6.

Answer: 0.4.

3. A game prize X is 0 dollars, 3 dollars or 12 dollars with probabilities 0.6, 0.3 and 0.1. [7 marks]

(a) Find its fair entry fee and prize variance. [3 marks]

E(X) = 3(0.3) + 12(0.1) = 2.1.

$E(X^2) = 9(0.3) + 144(0.1) = 17.1$; variance = $17.1 - 2.1^2$.

Answer: Fair fee 2.10 dollars; variance 12.69 dollars^2.

(b) A promotion gives twice the prize but charges 5 dollars. Find expected net gain and its variance. [2 marks]

Y = 2X − 5.

E(Y) = 4.2 − 5; Var(Y) = 4(12.69).

Answer: Mean −0.80 dollars; variance 50.76 dollars^2.

(c) Explain why doubling prizes does not necessarily benefit the player on average. [2 marks]

Here the fee 5 exceeds the new expected gross prize 4.2.

Answer: The increased fee may exceed the increase in expected prize.

4. A policy has a claim of 1000 dollars with probability 0.02 and no claim otherwise. Administration costs 8 dollars per policy. The premium c must be a whole number of dollars. [8 marks]

(a) Find the expected claim and the smallest premium giving expected profit at least 15 dollars. [3 marks]

Expected claim = 1000(0.02) = 20.

Expected profit = c − 8 − 20 ≥ 15, giving c ≥ 43.

Answer: Expected claim 20 dollars; minimum premium 43 dollars.

(b) At that premium, find the probability of a negative profit on a single policy. [2 marks]

Without a claim profit is 43 − 8 = 35; with a claim it is −965.

Only the claim event produces a loss.

Answer: 0.02.

(c) Find the standard deviation of single-policy profit. [3 marks]

Claim variance = $1000^2(0.02)(0.98) = 19600$.

Subtracting a claim and adding fixed charges preserves its SD: $\sqrt{19600} = 140$.

Answer: 140 dollars.

5. A usage count X is 0, 1 or 2 with probabilities 0.5, 0.3 and 0.2. A nonlinear cost is $C = 5 + 2X + X^2$ dollars. [7 marks]

(a) Construct the distribution of C. [2 marks]

Evaluate the cost separately at x = 0, 1 and 2.

Answer: C = 5, 8, 13 with probabilities 0.5, 0.3, 0.2.

(b) Find E(C) and Var(C). [3 marks]

$E(C) = 5(0.5) + 8(0.3) + 13(0.2) = 7.5$.

$E(C^2) = 25(0.5) + 64(0.3) + 169(0.2) = 65.5$.

Variance $= 65.5 - 7.5^2$.

Answer: Mean 7.50 dollars; variance 9.25 dollars^2.

(c) Why should the affine variance rule not be applied directly to the whole expression for C? [2 marks]

Construct the transformed distribution or calculate the necessary higher moments.

Answer: The X^2 term makes the transformation nonlinear.

6. Independent daily demands are A = 0 or 1 with probabilities 0.4 and 0.6, and B = 0 or 2 with probabilities 0.7 and 0.3. Set T = A + B. [7 marks]

(a) Find the distribution of T. [3 marks]

Multiply probabilities of the four independent pairs: (0,0), (1,0), (0,2), (1,2).

Answer: P(T = 0, 1, 2, 3) = 0.28, 0.42, 0.12, 0.18.

(b) Find E(T) and Var(T). [2 marks]

A has mean 0.6 and variance 0.24.

B has mean 0.6 and variance $4(0.3)(0.7) = 0.84$; add under independence.

Answer: Mean 1.2; variance 1.08.

(c) Would the probability table in (a) still follow from the marginal tables if independence were removed? [2 marks]

The products used for the joint pairs would no longer be justified.

Answer: No.

7. Use the distribution T = 0, 1, 2, 3 with probabilities 0.28, 0.42, 0.12, 0.18. A shortage occurs when T ≥ 2. Assume seven independent days follow this same distribution. [6 marks]

(a) Define a shortage indicator and give its mean and variance. [2 marks]

Shortage probability = 0.12 + 0.18 = 0.3.

A Bernoulli indicator has variance 0.3(0.7).

Answer: I = 1 for T ≥ 2, otherwise 0; mean 0.3 and variance 0.21.

(b) State the distribution, mean and variance of the number S of shortage days. [2 marks]

Seven fixed independent days have common shortage probability 0.3.

Use np and np(1 − p).

Answer: $S \sim \text{Bin}(7, 0.3)$; mean 2.1, variance 1.47.

(c) Find P(S ≥ 3). [2 marks]

Use 1 − P(S ≤ 2).

$\sum_{k=3}^{7} \binom{7}{k}(0.3)^k(0.7)^{7-k} = 0.3529.$

Answer: 0.3529

8. Each batch has one shared environment: equally likely low-risk or high-risk. Conditional on that environment, two items fail independently with probabilities 0.2 or 0.8 respectively. X counts failed items. [8 marks]

(a) Find each item's unconditional failure probability. [2 marks]

$0.5(0.2) + 0.5(0.8) = 0.5$.

Answer: 0.5.

(b) Construct the unconditional distribution of X. [3 marks]

$P(X = 2) = 0.5(0.2^2) + 0.5(0.8^2) = 0.34$.

By symmetry $P(X = 0) = 0.34$; the remaining probability is 0.32.

Answer: $P(X = 0) = 0.34$; $P(X = 1) = 0.32$; $P(X = 2) = 0.34$.

(c) Compare its variance with Bin(2, 0.5) and identify the failed assumption. [3 marks]

$E(X) = 1$; $E(X^2) = 0.32 + 4(0.34) = 1.68$.

The shared environment links the outcomes even though their marginal probabilities match.

Answer: Actual variance 0.68; binomial variance 0.5. Unconditional independence fails.

9. For independent trials with success probability 0.1, define X_n as the success count in n trials. A design requires $P(X_n \geq 5) \leq 0.05$. [8 marks]

(a) Evaluate the tail probability for n = 19, 20 and 21. [3 marks]

For each n use Bin(n, 0.1) and calculate $1 - P(X_n \leq 4)$.

Answer: 19: 0.0352; 20: 0.0432; 21: 0.0522.

(b) Which is the largest of these three acceptable n values? [2 marks]

The n = 20 probability is below 0.05, while n = 21 is above it.

Answer: n = 20.

(c) Explain why increasing n further cannot reduce this tail probability, when p stays fixed. [3 marks]

Couple the larger experiment with the first n trials: later trials only add 0 or more successes.

Therefore the event of at least five successes cannot become less likely.

Answer: Adding extra trials cannot reduce an existing success count.

10. A binomial model for a daily count has exact mean 6 and variance 4.5. An alert is raised for a count of at least 10. [7 marks]

(a) Find n and p. [3 marks]

$1 - p = 4.5/6 = 0.75$.

$p = 0.25$ and $n = 6/0.25 = 24$.

Answer: $n = 24$, $p = 0.25$.

(b) Find the probability of an alert. [2 marks]

Use $X \sim \text{Bin}(24, 0.25)$ and $P(X \geq 10) = 1 - P(X \leq 9)$.

$\sum_{k=10}^{24} \binom{24}{k}(0.25)^k(0.75)^{24-k} = 0.0547$.

Answer: 0.0547

(c) If the true success probability changes to 0.2, may the old alert probability be used unchanged? Explain. [2 marks]

The model parameters have changed.

Recalculate the Bin(24, 0.2) tail; old mean/variance and alert probabilities no longer apply.

Answer: No.