Discrete distributions - Applications and reasoning test

Name: _____ Date: _____

10 questions | 72 marks | 75 minutes

An integrated paper requiring distribution construction, nonlinear costs, model criticism and justified decisions.

Use a graphics calculator or appropriate digital tool for labelled tasks.

1. A count X takes values 0, 1 and 3 with probabilities a, b and 0.2. Its expected value is 1.1. [7 marks]

(a) Find a and b. [3 marks]

(b) Find Var(X). [2 marks]

(c) Find P(X ≤ 1) and state whether observing 3 is impossible. [2 marks]

2. An integer-valued X has F(0) = a, F(1) = 3a, F(2) = 1 − a and F(3) = 1. No other values occur. Also P(X = 1) = 0.4. [7 marks]

(a) Find a and the probability table. [3 marks]

(b) Find the mean and variance. [2 marks]

(c) Find P(1 < X ≤ 3). [2 marks]

3. A game prize X is 0 dollars, 3 dollars or 12 dollars with probabilities 0.6, 0.3 and 0.1. [7 marks]

(a) Find its fair entry fee and prize variance. [3 marks]

(b) A promotion gives twice the prize but charges 5 dollars. Find expected net gain and its variance. [2 marks]

(c) Explain why doubling prizes does not necessarily benefit the player on average. [2 marks]

4. A policy has a claim of 1000 dollars with probability 0.02 and no claim otherwise. Administration costs 8 dollars per policy. The premium c must be a whole number of dollars. [8 marks]

(a) Find the expected claim and the smallest premium giving expected profit at least 15 dollars. [3 marks]

(b) At that premium, find the probability of a negative profit on a single policy. [2 marks]

(c) Find the standard deviation of single-policy profit. [3 marks]

5. A usage count X is 0, 1 or 2 with probabilities 0.5, 0.3 and 0.2. A nonlinear cost is $C = 5 + 2X + X^2$ dollars. [7 marks]

(a) Construct the distribution of C. [2 marks]

(b) Find E(C) and Var(C). [3 marks]

(c) Why should the affine variance rule not be applied directly to the whole expression for C? [2 marks]

6. Independent daily demands are A = 0 or 1 with probabilities 0.4 and 0.6, and B = 0 or 2 with probabilities 0.7 and 0.3. Set T = A + B. [7 marks]

(a) Find the distribution of T. [3 marks]

(b) Find E(T) and Var(T). [2 marks]

(c) Would the probability table in (a) still follow from the marginal tables if independence were removed? [2 marks]

7. Use the distribution T = 0, 1, 2, 3 with probabilities 0.28, 0.42, 0.12, 0.18. A shortage occurs when T ≥ 2. Assume seven independent days follow this same distribution. [6 marks]

(a) Define a shortage indicator and give its mean and variance. [2 marks]

(b) State the distribution, mean and variance of the number S of shortage days. [2 marks]

(c) Find P(S ≥ 3). [2 marks]

8. Each batch has one shared environment: equally likely low-risk or high-risk. Conditional on that environment, two items fail independently with probabilities 0.2 or 0.8 respectively. X counts failed items. [8 marks]

(a) Find each item’s unconditional failure probability. [2 marks]

(b) Construct the unconditional distribution of X. [3 marks]

(c) Compare its variance with Bin(2, 0.5) and identify the failed assumption. [3 marks]

9. For independent trials with success probability 0.1, define X_n as the success count in n trials. A design requires $P(X_n \geq 5) \leq 0.05$. [8 marks]

(a) Evaluate the tail probability for n = 19, 20 and 21. [3 marks]

(b) Which is the largest of these three acceptable n values? [2 marks]

(c) Explain why increasing n further cannot reduce this tail probability, when p stays fixed. [3 marks]

10. A binomial model for a daily count has exact mean 6 and variance 4.5. An alert is raised for a count of at least 10. [7 marks]

(a) Find n and p. [3 marks]

(b) Find the probability of an alert. [2 marks]

(c) If the true success probability changes to 0.2, may the old alert probability be used unchanged? Explain. [2 marks]