Proof and circle geometry - Applications test

10 questions | 56 marks | 75 minutes

Use logical claims, exact arithmetic and circle theorems to analyse practical designs and identify errors.

Equivalent correct methods earn the corresponding marks.

1. A numerical checker accepts an integer exactly when it is divisible by both 3 and 5. [5 marks]

(a) Express the rule as an equivalence using divisibility by one integer. [3 marks]

3 and 5 are distinct primes, so their product divides any integer divisible by both. A multiple of 15 is visibly divisible by both.

Answer: Accepted $\Leftrightarrow$ divisible by 15.

(b) Does acceptance imply divisibility by 30? Give evidence. [2 marks]

15 is accepted but is not divisible by 30.

Answer: No.

2. An algorithm assumes $a < b$ always implies $a^2 < b^2$. [5 marks]

(a) Give a counterexample using negative integers. [2 marks]

$-3 < -2$ but $9 > 4$.

Answer: $a = -3$, $b = -2$.

(b) Prove the implication is valid under the extra condition $0 \leq a < b$. [3 marks]

$b^2 - a^2 = (b - a)(b + a)$. Both factors are positive, so $b^2 > a^2$.

Answer: It is valid on this domain.

3. A student says: “If x+y is rational, then x and y must each be rational.” [5 marks]

(a) Disprove this implication. [2 marks]

Both terms are irrational but their sum is 0.

Answer: $x = \sqrt{2}$, $y = -\sqrt{2}$.

(b) Prove that if x is rational and x+y is rational, then y is rational. [3 marks]

$y = (x + y) - x$ is the difference of two rational numbers.

Answer: It follows by subtraction.

4. Prove that $\sqrt{3}$ is irrational. [7 marks]

(a) Show an integer not divisible by 3 has a square not divisible by 3. [3 marks]

The integer is $3k + 1$ or $3k + 2$; squaring gives remainder 1 modulo 3 in either case.

Answer: Its square has remainder 1.

(b) Use a fraction in lowest terms to complete a contradiction proof. [4 marks]

Assume $\sqrt{3} = a/b$ in lowest terms. Then $a^2 = 3b^2$, so part (a) forces $a = 3k$.

Substitution gives $b^2 = 3k^2$, so 3 divides b as well, contradicting lowest terms.

Answer: No rational fraction equals $\sqrt{3}$.

5. A formula $f(x) = 1/(x - 2)$ is to be used only when $-1 \le x < 5$. [5 marks]

(a) Write its permitted input set in interval notation. [3 marks]

Exclude x=2 because its denominator is zero; keep the original outside bounds.

Answer: $[-1, 2) \cup (2, 5)$.

(b) Explain why $[-1, 5)$ alone is incorrect. [2 marks]

At x=2 the formula is undefined.

Answer: It includes the forbidden input 2.

6. Two camera positions P and Q lie on opposite segments of chord AB of a circular arena. The minor central angle AOB is 84°. P lies on the major arc and Q on the minor arc. [5 marks]

(a) Find the viewing angle APB. [2 marks]

P intercepts the minor arc: $84/2 = 42$.

Answer: 42°.

(b) Find AQB and explain why the two angles differ. [3 marks]

Q intercepts the major arc of 276°; halve it to get 138°. The vertices lie in opposite segments.

Answer: 138°.

7. A four-sided frame ABCD is designed to fit a circular opening. In cyclic order its angles A and C are $2x + 10$ and $3x - 5$ degrees, and B is 110°. [6 marks]

(a) Find x, A and C. [3 marks]

$2x + 10 + 3x - 5 = 180$ gives $5x = 175$.

Answer: $x = 35$; $\text{A} = 80^\circ$, $\text{C} = 100^\circ$.

(b) Find the exterior angle at D formed by extending one side. [3 marks]

$\text{D} = 180 - 110 = 70^\circ$; its exterior angle is $180 - 70 = 110^\circ$, equal to opposite B.

Answer: 110°.

8. A circular pond has radius 9 m. A post P is 15 m from its centre. Two straight cables from P just touch the pond at A and B. [5 marks]

(a) Find the length of cable PA. [3 marks]

The radius to A is perpendicular to PA. $PA = \sqrt{15^2 - 9^2} = 12$.

Answer: 12 m.

(b) Find the total length of the two cables and justify the equality you use. [2 marks]

Tangents PA and PB from the same external point are equal, so the total is 2×12.

Answer: 24 m.

9. A survey line from P crosses a circle at A and B, with PA=6 m and AB=18 m. A tangent from P touches at T. [6 marks]

(a) Find PT. [3 marks]

PB=24; $PT^2 = 6 \times 24 = 144$.

Answer: 12 m.

(b) A second survey line has outside portion 9 m. Find its portion inside the circle. [3 marks]

The full length L satisfies $9L = 144$, giving L=16; subtract 9.

Answer: 7 m.

10. A circular skylight has centre O and a non-diameter chord AB. M is the midpoint of AB. [7 marks]

(a) Prove OM is perpendicular to AB. [4 marks]

OA=OB are radii, AM=BM from midpoint, and OM is shared.

The equal angles OMA and OMB lie on a straight line, so both are 90°.

Answer: Triangles OMA and OMB are congruent by SSS.

(b) If the radius is 10 cm and AB is 16 cm, find OM. [3 marks]

AM=8 and $OM^2 = 10^2 - 8^2 = 36$.

Answer: 6 cm.