Proof and circle geometry - Reasoning test

10 questions | 52 marks | 75 minutes

Explain why methods work, judge the information in a diagram, and combine algebra with circle properties.

Equivalent correct methods earn the corresponding marks.

1. For an integer n, compare “n is divisible by 12” with “n is divisible by 6”. [6 marks]

(a) Is divisibility by 12 necessary, sufficient, both or neither for divisibility by 6? [3 marks]

$12k = 6(2k)$ proves sufficiency. The value n=6 disproves necessity.

Answer: Sufficient but not necessary.

(b) Complete and justify: n is divisible by 12 if and only if n is divisible by 3 and ___. [3 marks]

The distinct prime-power factors 3 and 4 have no common factor, so their combined divisibility is divisibility by 12.

Answer: 4.

2. For positive integers n, consider $n^2 + n + 17$. [4 marks]

(a) Find a positive-integer value for which this expression is not prime. [2 marks]

$17^2 + 17 + 17 = 17(17 + 2)$.

Answer: $n = 17$, giving $323 = 17 \times 19$.

(b) Explain the logical effect of this one result, even if many earlier values were prime. [2 marks]

A single admissible composite output is enough.

Answer: It disproves the universal prime claim.

3. Let n be any integer. [6 marks]

(a) Prove $n^2 + n$ is even without listing individual n values. [3 marks]

Factor to $n(n+1)$; one of the consecutive integers is even.

Answer: It is even.

(b) Prove that if n is odd, then n^2 is odd. [3 marks]

Write $n = 2k + 1$. Then $n^2 = 2(2k^2 + 2k) + 1$.

Answer: It is odd.

4. Consider $\sqrt{2}$ and $\sqrt{8}$. [5 marks]

(a) Simplify their sum and justify that it is irrational. [3 marks]

$\sqrt{8} = 2\sqrt{2}$, so the sum is $3\sqrt{2}$. If this were rational, division by 3 would make $\sqrt{2}$ rational.

Answer: $3\sqrt{2}$ is irrational.

(b) Find their product and state whether that product is rational. [2 marks]

$\sqrt{2}\sqrt{8} = \sqrt{16} = 4$.

Answer: 4, rational.

5. Let $A = (-\infty, -1] \cup [3, \infty)$ and $B = (-4, 5)$. [5 marks]

(a) Find $A \cap B$. [3 marks]

Keep only points of each portion of A that also lie between -4 and 5.

Answer: $(-4, -1] \cup [3, 5)$.

(b) Which of -4,-1,3,5 belong to the intersection? [2 marks]

B excludes -4 and 5; A includes -1 and 3.

Answer: -1 and 3 only.

6. A, B, C, D are distinct points in that cyclic order on a circle. Angle ABD is 32° and angle BAD is 68°. [4 marks]

(a) Find angle ACD. [2 marks]

ABD and ACD subtend chord AD from the same segment.

Answer: 32°.

(b) Find angle BCD. [2 marks]

BAD and BCD are opposite angles in cyclic ABCD, so $180 - 68 = 112$.

Answer: 112°.

7. A convex quadrilateral has adjacent angles $A = 70^\circ$ and $B = 110^\circ$. [5 marks]

(a) Does this information alone prove the quadrilateral is cyclic? Give a reason or a counterexample. [3 marks]

A parallelogram with angles 70,110,70,110 degrees has the same adjacent angles but is not cyclic: opposite angles are not supplementary.

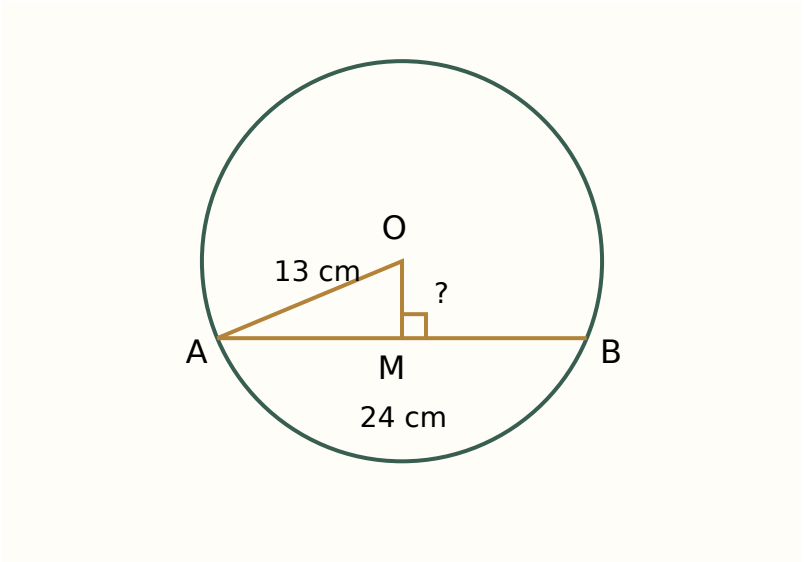
Answer: No.

(b) If C is also 110°, prove the quadrilateral is cyclic. [2 marks]

The relevant pair A,C is opposite, not adjacent.

Answer: $A + C = 180^\circ$, so the converse applies.

8. A circle has radius 13 cm. Chord AB is 24 cm long and chord CD is 10 cm long. [6 marks]



(a) Find each chord's distance from the centre. [3 marks]

For AB, $d^2 = 169 - 12^2 = 25$. For CD, $d^2 = 169 - 5^2 = 144$.

Answer: AB: 5 cm; CD: 12 cm.

(b) Explain generally why the longer of two chords in one circle lies closer to its centre. [3 marks]

From $d^2 = r^2 - a^2$ at fixed radius r, increasing positive half-length a decreases d.

Answer: Its half-length is larger, leaving a smaller perpendicular distance.

9. Two chords meet at E. On AB, AE=2x and EB=x; on CD, CE=4 and ED=8. All lengths use the same unit. [5 marks]

(a) Find x and the length AB. [3 marks]

$2x^2 = 4 \times 8 = 32$, so $x^2 = 16$. Choose positive x=4; AB=3x=12.

Answer: $x = 4$ and AB=12.

(b) Compare the distances of AB and CD from the centre. [2 marks]

CD=4+8=12, equal to AB. Equal chords in one circle are equally distant from its centre.

Answer: They are equal.

10. Triangles ABC and DEF have respective side lengths AB=8, BC=10, AC=12 and DE=12, EF=15, DF=18. [6 marks]

(a) Prove they are similar, specifying the correspondence. [3 marks]

$AB/DE = BC/EF = AC/DF = 2/3$.

Answer: $\triangle ABC \sim \triangle DEF$ by proportional SSS.

(b) Find the area ratio DEF:ABC without needing to calculate either absolute area. [3 marks]

The corresponding length scale from ABC to DEF is $3/2$.

Areas scale by the square: $(3/2)^2 = 9/4$.

Answer: $9:4$.