Proof and circle geometry - Challenge test

10 questions | 61 marks | 75 minutes

A final synthesis paper requiring careful domains, valid converses and multi-step geometric reasoning.

Equivalent correct methods earn the corresponding marks.

1. A rectangle has positive side lengths a and b. [7 marks]

(a) Prove its two diagonals have equal lengths. [3 marks]

Each diagonal is the hypotenuse of a right triangle with legs a and b.

Answer: Both diagonals have length $\sqrt{a^2 + b^2}$.

(b) Is the converse “equal diagonals imply a rectangle” true for all quadrilaterals? Give a checked counterexample. [4 marks]

Take vertices (0,0),(4,0),(3,2),(1,2). Both diagonals have length $\sqrt{13}$, but the sloping sides are not perpendicular to the horizontal bases.

Answer: No.

2. Work over the integers. [6 marks]

(a) Prove an odd integer has an odd square. [2 marks]

$(2k + 1)^2 = 2(2k^2 + 2k) + 1$.

Answer: It does.

(b) Hence prove the equivalence: n is even if and only if n^2 is even. [4 marks]

If n=2k then $n^2 = 2(2k^2)$ is even.

If n^2 is even but n were odd, part (a) would make n^2 odd, a contradiction.

Answer: Both directions hold.

3. Investigate operations on irrational numbers. [6 marks]

(a) Give two irrational numbers whose sum and product are both rational. [2 marks]

Their sum is 0 and product -2.

Answer: $\sqrt{2}$ and $-\sqrt{2}$.

(b) Give two irrational numbers with rational sum but irrational product, and justify both claims. [4 marks]

Both are irrational; their sum is 2.

The product is $2\sqrt{2}-2$, irrational because a rational value would make $\sqrt{2}$ rational.

Answer: $\sqrt{2}$ and $2-\sqrt{2}$.

4. Compare two number systems. [6 marks]

(a) Prove there is no least positive rational number. [3 marks]

If q were least and positive, q/2 would be a smaller positive rational, a contradiction.

Answer: No such rational exists.

(b) Explain why this argument does not show there is no least positive integer. [3 marks]

In particular, 1/2 is not an integer. The positive integers do have least member 1.

Answer: Dividing an integer by 2 need not leave an integer.

5. Solve $\frac{x-1}{x+2} \geq 0$ for real x. [6 marks]

(a) Identify the values where the sign can change and the excluded input. [2 marks]

The numerator is zero at 1 and denominator at -2.

Answer: Critical values -2 and 1; exclude -2.

(b) Use a sign analysis to write the full interval solution. [4 marks]

For x<-2 both factors are negative, giving a positive ratio.

For -2<x<1 the ratio is negative; for x>1 it is positive. Include zero at x=1, but never x=-2.

Answer: $(-\infty, -2) \cup [1, \infty)$.

6. For chord AB, a minor central angle is $3x + 6$ degrees. An inscribed angle on the major arc AB is $x + 13$ degrees. [5 marks]

(a) Find x and both angles. [3 marks]

$3x + 6 = 2(x + 13)$ gives x=20.

Answer: $x = 20$; central 66°, inscribed 33°.

(b) Find the inscribed angle from a point on the minor arc AB. [2 marks]

It intercepts the major arc of $360 - 66 = 294$ degrees; halve to get 147.

Answer: 147°.

7. ABCD is cyclic in that order. Its opposite angles A and C are $x + 30$ and $2x$ degrees. Its opposite angles B and D are $2y - 10$ and $y + 40$ degrees. [7 marks]

(a) Find x,y and all four interior angles. [4 marks]

$3x + 30 = 180$ and $3y + 30 = 180$. Substitute to find each angle.

Answer: $x = y = 50$; A=80, B=90, C=100, D=90 degrees.

(b) Which diagonal is a diameter? Justify it. [3 marks]

Angle ABC is 90°, so the arc AC not containing B is a semicircle; AC is a diameter.

Answer: AC.

8. A triangle ABC is inscribed in a circle. Angle BAC=58° and angle ACB=36°. A tangent is drawn at A. [5 marks]

(a) Find angle ABC. [2 marks]

$180 - 58 - 36 = 86$.

Answer: 86°.

(b) Find the acute angle between the tangent and chord AC, naming the relevant theorem. [3 marks]

The alternate-segment theorem matches that tangent-chord angle with angle ABC.

Answer: 86°.

9. A tangent from P is 6 cm long. A secant from P has outside length x cm and inside length 9 cm. [6 marks]

(a) Find x, rejecting any invalid root. [3 marks]

$x(x+9)=36$, so $(x+12)(x-3)=0$. Reject -12 as a length.

Answer: $x=3$ cm.

(b) A second secant from P has outside length 4 cm. Find its inside length. [3 marks]

Its whole length L satisfies $4L=36$, so L=9 and inside length is $9-4=5$.

Answer: 5 cm.

10. Distinct points A,B,C,D lie in that order on a circle. Chords AC and BD intersect at E inside the circle. [7 marks]

(a) Prove triangles AEB and DEC are similar and deduce $AE\cdot EC=BE\cdot ED$. [5 marks]

Angles AEB and DEC are vertically opposite.

Angles ABE and DCE equal angles ABD and DCA; both subtend chord AD in the same segment.

Thus AEB corresponds to DEC. From $AE/DE=BE/CE$, cross-multiply.

Answer: They are similar by AA, giving the stated product.

(b) If AE=6 cm, EC=10 cm and BE=4 cm, find ED. [2 marks]

$6\times10=4\times ED$, so ED=15.

Answer: 15 cm.