Further trigonometry - Synthesis test 5

10 questions | 49 marks | 75 minutes

A more demanding paper combining reasoning, error analysis, exact values and model construction. Explain exclusions and check the meaning of every result.

Equivalent correct methods earn the corresponding marks.

1. A student calculates the length of an arc of radius 5 cm and angle 30° as 150 cm. [4 marks]

(a) Find the correct length and explain the student error. [2 marks]

Convert 30° to $\pi/6$, then use $5(\pi/6)$.

Answer: $5\pi/6$ cm; the student used degrees in the radian formula.

(b) Find $\sin(-5\pi/6)$ and $\cos(-5\pi/6)$ exactly. [2 marks]

The point lies in quadrant III after adding 2π; reference angle $\pi/6$.

Answer: $-1/2$ and $-\sqrt{3}/2$.

2. A graph has minimum -2 at $x = 1$, maximum 6 at $x = 5$, and the next minimum at $x = 9$. [5 marks]

(a) Construct a cosine rule that matches these observations. [3 marks]

Midline 2, amplitude 4 and period 8; negative cosine starts at the minimum at 1.

Answer: $y = 2 - 4\cos(\frac{\pi}{4}(x - 1))$.

(b) Find the two midline crossings between the consecutive minima. [2 marks]

They occur a quarter and three quarters of the period after 1.

Answer: $x = 3, 7$.

3. Solve $\cos 2x = \cos x$ on $[0, 2\pi)$. [5 marks]

(a) Find every solution algebraically. [3 marks]

Use $2\cos^2 x - 1 = \cos x$.

Factor $(2\cos x + 1)(\cos x - 1) = 0$.

Answer: $0, 2\pi/3, 4\pi/3$.

(b) Explain why dividing first by $\cos x$ is an unnecessarily risky method, even though its zeros are not solutions here. [2 marks]

At cosine zero the original sides are -1 and 0, so those inputs fail; factoring handles the equation directly without extra domain work.

Answer: Division needs a separate zero check and produces a reciprocal expression.

4. An angle A is in quadrant II and $\sin A = 3/5$. [5 marks]

(a) Find $\sin(A + \pi/6)$ and $\cos(A + \pi/6)$ exactly. [3 marks]

Use $\cos A = -4/5$ and the standard values at $\pi/6$.

Apply the sine and cosine sum formulas.

Answer: $(3\sqrt{3} - 4)/10$ and $-(4\sqrt{3} + 3)/10$.

(b) Identify the quadrant of $A + \pi/6$ using those signs. [2 marks]

$3\sqrt{3} > 4$, so sine is positive; cosine is negative.

Answer: Quadrant II.

5. An angle θ lies in quadrant IV with $\sin\theta = -4/5$ and $0 \le \theta < 2\pi$. [5 marks]

(a) Find $\cos(\theta/2)$ exactly. [3 marks]

Cosine of theta is 3/5.

Half-angle cosine square is $(1 + 3/5)/2 = 4/5$.

The half-angle is in quadrant II, so choose the negative root.

Answer: $-2/\sqrt{5}$ or $-2\sqrt{5}/5$.

(b) Find $\tan(\theta/2)$ exactly. [2 marks]

Half-angle sine is positive $1/\sqrt{5}$; divide by $-2/\sqrt{5}$.

Answer: $-1/2$.

6. A combined signal is $f(x) = a\sin x + b\cos x$, with $a > 0$, amplitude 10 and $f(0) = -6$. [5 marks]

(a) Find a, b and express the signal as one shifted sine. [3 marks]

At 0, $b = -6$; then $a^2 = 100 - 36 = 64$ and $a > 0$.

Match cosine phase 4/5 and sine phase 3/5.

Answer: $a = 8$, $b = -6$; $f(x) = 10\sin(x - \phi)$, $\phi = \tan^{-1}(3/4)$.

(b) Find the smallest positive zero of the signal. [2 marks]

Set the shifted sine argument to 0; this is a positive angle below $\pi/2$.

Answer: $x = \tan^{-1}(3/4)$.

7. Find intersections of $y = \sec x$ and $y = \csc x$ on $[0, 2\pi)$. [5 marks]

(a) Find the input values, stating the domain restrictions used. [3 marks]

Cross-multiplication is valid on that common domain and gives $\sin x = \cos x$.

Both selected inputs satisfy the original restrictions.

Answer: $x = \pi/4, 5\pi/4$; require nonzero sine and cosine.

(b) Give the two intersection coordinates. [2 marks]

Take the reciprocal of the sine/cosine values $\pm\sqrt{2}/2$.

Answer: $(\pi/4, \sqrt{2})$ and $(5\pi/4, -\sqrt{2})$.

8. A student writes $\sqrt{1-\sin^2 x}=\cos x$ for every real x. [4 marks]

(a) Correct the statement and explain it. [2 marks]

The inside is $\cos^2 x$ and the principal square root is nonnegative.

Answer: $\sqrt{1-\sin^2 x}=|\cos x|$.

(b) Give a counterexample to the student statement using an exact angle. [2 marks]

Both sides are defined, so this single failure disproves the universal claim.

Answer: At $x=\pi$, the left is 1 and the right -1.

9. An ideal tide repeats every 8 hours. Its height is 1 m at $t=0$ and 5 m at $t=4$. Assume these are a minimum and the next maximum. [6 marks]

(a) Construct a model and calculate its height after 2 hours. [3 marks]

Midline 3, amplitude 2, period 8 and starting minimum determine the rule.

At 2 hours the argument is $\pi/2$.

Answer: $h(t)=3-2\cos(\pi t/4)$; $h(2)=3$ m.

(b) Find the duration for which the tide is at least 4 m in one 8-hour cycle. [3 marks]

Cosine must be at most $-1/2$, so $8/3\le t\le 16/3$.

Subtract the bounds.

Answer: 8/3 hours.

10. Define $g(x)=1+2\sin x\cos x$. [5 marks]

(a) Find its exact range and period. [2 marks]

Rewrite as $1+\sin 2x$.

Answer: Range $[0,2]$; period π.

(b) Solve $g(x)=1/2$ on $[0,2\pi)$. [3 marks]

Solve $\sin 2x=-1/2$ over $0\le 2x<4\pi$.

The four argument values are $7\pi/6, 11\pi/6, 19\pi/6, 23\pi/6$; divide by 2.

Answer: $7\pi/12, 11\pi/12, 19\pi/12, 23\pi/12$.