Vectors in two dimensions - Practice test 3

10 questions | 52 marks | 70 minutes

Connect exact values, projections and geometric structure, with a short proof to finish.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{v} = (4\sqrt{3}, -4)$. [4 marks]

(a) Find its magnitude and direction from the positive x-axis. [2 marks]

Magnitude $\sqrt{48 + 16} = 8$.

The reference angle is 30° in quadrant IV.

Answer: 8 and 330°.

(b) Find the unit vector in the same direction. [2 marks]

Divide each component by eight.

Answer: $(\sqrt{3}/2, -1/2)$.

2. In an original coordinate system A=(-3,2) and B=(1,5). A new origin is placed at old coordinates (2,-1), with axes pointing in the same directions. [5 marks]

(a) Give the new coordinates of A and B. [2 marks]

Subtract the old coordinates of the new origin from each point.

Answer: A'=(-5,3), B'=(-1,6).

(b) Calculate AB in both systems and explain the agreement. [3 marks]

Old: $(1 + 3, 5 - 2)$; new: $(-1 + 5, 6 - 3)$.

A common origin shift cancels from a difference.

Answer: $(4, 3)$ in both systems.

3. A triangular route starts with displacement a=(2,1) km, then b=(-1,4) km, then c back to the starting point. [5 marks]

(a) Find c. [2 marks]

A closed route has $a + b + c = 0$; negate $(1, 5)$.

Answer: $(-1, -5)$ km.

(b) Find the exact total distance travelled. [3 marks]

Add the three segment magnitudes, not the vectors.

Answer: $\sqrt{5} + \sqrt{17} + \sqrt{26}$ km.

4. Let u=(2,1), v=(t,3) and w=(4,s). [4 marks]

(a) Find t if v is parallel to u. [2 marks]

The y-component multiplier is three; apply it to x.

Answer: $t = 6$.

(b) Find s if w is perpendicular to u. [2 marks]

$u \cdot w = 8 + s = 0$.

Answer: $s = -8$.

5. A=(1,0), B=(5,2), C=(3,6). [6 marks]

(a) Find angle BAC exactly. [3 marks]

AB=$(4, 2)$, AC=$(2, 6)$; dot product 20.

The cosine is $20/(\sqrt{20}\sqrt{40}) = 1/\sqrt{2}$.

Answer: 45°.

(b) Use a second vector calculation to classify the triangle. [3 marks]

BA·BC=$(-4, -2) \cdot (-2, 4) = 0$.

AB and BC both have length $\sqrt{20}$.

Answer: An isosceles right-angled triangle, right angle at B.

6. A force u=(-4,1) N is resolved along the direction v=(3,0). [4 marks]

(a) Find its signed scalar projection and vector projection along v. [2 marks]

Dot product -12, direction length 3; divide for the scalar.

Multiply the unit direction $(1, 0)$ by -4.

Answer: Scalar -4 N; vector $(-4, 0)$ N.

(b) Find the perpendicular component. [2 marks]

Subtract $(-4, 0)$ from the force.

Answer: $(0, 1)$ N.

7. A line passes through O and has direction v=(2,1). P has position u=(5,3). [6 marks]

(a) Find the position of the perpendicular foot H on the line. [3 marks]

Projection coefficient is $(10 + 3)/(4 + 1) = 13/5$.

Answer: $H = (26/5, 13/5)$.

(b) Find the exact shortest distance from P to the line. [3 marks]

PH as a difference is $(-1/5, 2/5)$ up to direction.

Its magnitude is $\sqrt{1/25 + 4/25} = 1/\sqrt{5}$.

Answer: $1/\sqrt{5}$.

8. The angle between $(a, 1)$ and $(1, 1)$ is 45°. [6 marks]

(a) Form an equation involving a using the dot product. [3 marks]

Set $(a + 1)/(\sqrt{a^2 + 1}\sqrt{2}) = 1/\sqrt{2}$ and simplify.

Answer: $a + 1 = \sqrt{a^2 + 1}$.

(b) Find a and check the original angle condition. [3 marks]

Squaring gives $a^2 + 2a + 1 = a^2 + 1$, hence $a = 0$.

The original left and right sides are both one; $(0, 1)$ makes 45° with $(1, 1)$.

Answer: $a = 0$.

9. A parallelogram has adjacent sides a=(5,0) and b=(3,4). [6 marks]

(a) Show that it is a rhombus and that its diagonals are perpendicular. [3 marks]

Calculate side lengths.

Calculate $(8,4)\cdot(2,-4)=16-16$.

Answer: Both sides have length 5; diagonals $(8,4)$ and $(2,-4)$ have dot product zero.

(b) Find its area using half the product of the diagonal lengths. [3 marks]

Diagonal lengths are $4\sqrt{5}$ and $2\sqrt{5}$.

Area=$\frac{1}{2}(4\sqrt{5})(2\sqrt{5})=20$.

Answer: 20 square units.

10. Non-zero vectors a and b satisfy $|a|=|b|=L$ and $a\cdot b=L^2/2$. O, A and B have position vectors 0, a and b. [6 marks]

(a) Prove that OAB is equilateral. [3 marks]

Every side has length L, and the included angle is 60°, so the triangle is non-degenerate.

Answer: $|b-a|^2=L^2+L^2-2(L^2/2)=L^2$.

(b) If M is the midpoint of AB, find the length OM. [3 marks]

OM=$(a+b)/2$.

Its squared magnitude is $(L^2+L^2+2L^2/2)/4=3L^2/4$.

Answer: $\sqrt{3}L/2$.