Vectors in two dimensions - Practice test 5

10 questions | 61 marks | 75 minutes

A more demanding synthesis paper combining unknowns, modelling, projections and general proofs.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$. [4 marks]

(a) Find $3\mathbf{a} + 2\mathbf{b}$ in component form. [2 marks]

$3a = (6, -3)$ and $2b = (-2, 6)$; add.

Answer: $(4, 3)$.

(b) Find a unit vector in this resultant's direction. [2 marks]

The resultant has length five, so divide by five.

Answer: $(4/5, 3/5)$.

2. A=(k,2), B=(5,m). The displacement BA is $(-3, 4)$. [4 marks]

(a) Find k and m. [2 marks]

BA=$(k - 5, 2 - m)$; compare the two components.

Answer: $k = 2$, $m = -2$.

(b) Find the exact length of AB and a unit vector from A to B. [2 marks]

AB is the opposite displacement $(3, -4)$.

Normalise by its length five.

Answer: Length 5; unit vector $(3/5, -4/5)$.

3. A non-zero vector v is parallel to (3,4), has magnitude 20 and has a negative x-component. [6 marks]

(a) Find v. [3 marks]

Write v=$t(3,4)$; $5|t| = 20$, so $t = \pm 4$.

A negative x-component requires $t = -4$.

Answer: $(-12, -16)$.

(b) Find its direction from the positive x-axis to 0.1°. [3 marks]

It lies in quadrant III.

Add 180° to $\tan^{-1}(4/3) = 53.130\ldots^\circ$.

Answer: 233.1°.

4. Non-zero vectors a and b satisfy $|a| = 2$, $|b| = 3$ and $a \cdot b = 3$. [6 marks]

(a) Find their included angle. [2 marks]

The cosine is $3/(2 \times 3) = 1/2$.

Answer: 60°.

(b) Find $|2\mathbf{a} - \mathbf{b}|$ and the scalar projection of a onto b. [4 marks]

The squared length is $4(4) - 4(3) + 9 = 13$.

The scalar projection is $(a \cdot b)/|b| = 3/3$.

Answer: $\sqrt{13}$; 1.

5. A=(-1,2), B=(3,4), C=(6,-2). D completes the parallelogram ABCD in that order. [7 marks]

(a) Find D. [3 marks]

Use $d = a + c - b = (-1, 2) + (6, -2) - (3, 4)$.

Answer: $(2, -4)$.

(b) Prove that the parallelogram is a rectangle and find its area. [4 marks]

AB=$(4, 2)$ and BC=$(3, -6)$ have dot product $12 - 12 = 0$.

Their lengths are $\sqrt{20}$ and $\sqrt{45}$; their product is thirty.

Answer: It is a rectangle; area 30 square units.

6. A vector u is perpendicular to (1,2), and its vector projection onto the positive x-axis is (-6,0). [5 marks]

(a) Find u. [3 marks]

The projection gives x=-6.

Perpendicularity gives $-6 + 2y = 0$, so y=3.

Answer: $(-6, 3)$.

(b) Find its scalar projection onto (0,-1). [2 marks]

Dot $(-6, 3)$ with the supplied unit direction $(0, -1)$.

Answer: -3.

7. Two adjustable forces are $m(3, 4)$ N and $n(2, -2)$ N, where m,n are positive. Their resultant must be (14,0) N. [7 marks]

(a) Find m and n. [4 marks]

Vertical equation: $4m - 2n = 0$, so $n = 2m$.

Horizontal equation: $3m + 2n = 14$, so $7m = 14$.

Answer: $m = 2$, $n = 4$.

(b) Compare the sum of the force magnitudes with the resultant magnitude. [3 marks]

The first force has magnitude $2(5) = 10$.

The second has magnitude $4\sqrt{8} = 8\sqrt{2}$.

Directions prevent magnitudes adding directly.

Answer: Sum $10 + 8\sqrt{2}$ N; resultant 14 N, so they are not equal.

8. Let u=(1,t) and v=(t,1), where t is real. [7 marks]

(a) Find all t making u and v parallel, and give the corresponding angles. [4 marks]

A scalar multiple exists when $1 - t^2 = 0$.

At t=1 the vectors are equal; at t=-1 they are opposites.

Answer: $t = 1$: 0°; $t = -1$: 180°.

(b) Find all t making u and v perpendicular. [3 marks]

Their dot product is $t + t = 2t$.

At zero the vectors $(1, 0)$ and $(0, 1)$ are non-zero.

Answer: $t = 0$.

9. A line through O has direction v=(3,4). P has position p=(-4,3). A point Q on the line has position $t\mathbf{v}$. [7 marks]

(a) Show that $|\overrightarrow{PQ}|^2 = 25 + 25t^2$. [4 marks]

Calculate $v \cdot p = -12 + 12 = 0$; both squared magnitudes are 25.

Expand the squared magnitude.

Answer: $|t\mathbf{v} - \mathbf{p}|^2 = 25t^2 - 2t(0) + 25$.

(b) Hence find the closest point on the line to P and the shortest distance, without calculus. [3 marks]

Since $25t^2 \geq 0$, the minimum squared distance is25 at t=0.

Take its square root.

Answer: The point is O; shortest distance 5.

10. In a quadrilateral ABCD, M,N,P,Q are the midpoints of AB,BC,CD,DA respectively. Assume the midpoint quadrilateral is non-degenerate. [8 marks]

(a) Use position vectors to prove that MNPQ is a parallelogram. [4 marks]

Write each midpoint as the average of its two endpoint position vectors.

Subtract to obtain both pairs of equal directed opposite sides.

Answer: MN=QP=$\frac{1}{2}(c-a)$ and NP=MQ=$\frac{1}{2}(d-b)$.

(b) Prove that perpendicular diagonals AC and BD make MNPQ a rectangle. [4 marks]

Their dot product is one quarter of $(c-a)\cdot(d-b) = 0$.

The non-degenerate parallelogram therefore has a right angle.

Answer: Adjacent side directions MN and NP are half AC and half BD.