Matrices and transformations - Practice test 2

10 questions | 62 marks | 70 minutes

Geometry and matrix models: Use matrices for cost changes, geometric mappings, projection, rotations and output constraints.

Equivalent correct methods earn the corresponding marks.

1. A store records notebook, folder and pen quantities in $N = \begin{pmatrix} 6 & 4 & 3 \\ 2 & 5 & 7 \end{pmatrix}$, one row per day. Prices are $p = (4, 3, 2)^T$ dollars. [6 marks]

(a) Calculate Np and interpret it. [3 marks]

Day 1 costs 24+12+6=42.

Day 2 costs 8+15+14=37.

Answer: $(42, 37)^T$ dollars.

(b) Prices change by $(1, 0, -0.5)^T$ dollars. Find the change in each daily total. [3 marks]

Multiply N by the price-change vector.

The daily changes are 6-1.5 and 2-3.5.

Answer: $(4.5, -1.5)^T$ dollars.

2. A linear map sends $(1, 1)$ to $(3, 1)$ and $(1, -1)$ to $(1, 3)$. [7 marks]

(a) Find the images of the two coordinate unit vectors and hence the matrix. [4 marks]

Add the supplied vector equations and divide by 2.

Subtract them and divide by 2.

Answer: $Ae_1 = (2, 2)^T$, $Ae_2 = (1, -1)^T$; $A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$.

(b) Find the image of $(2, 3)$ and the area multiplier. [3 marks]

Multiply by A.

The determinant is -4; take its absolute value.

Answer: $(7, 1)$; multiplier 4.

3. Let $F=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $D=\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$. [6 marks]

(a) Calculate FD and DF. [4 marks]

Keep factor order in the row-column products.

Answer: $FD=\begin{pmatrix}0 & 1\\ 3 & 0\end{pmatrix}$, $DF=\begin{pmatrix}0 & 3\\ 1 & 0\end{pmatrix}$.

(b) Apply both products to $(2,1)$ and explain the different geometric orders. [2 marks]

FD stretches horizontally then swaps coordinates.

DF swaps coordinates then stretches horizontally.

Answer: FD gives $(1,6)$; DF gives $(3,2)$.

4. A point is rotated 60° counterclockwise about the origin. [6 marks]

(a) Write the exact rotation matrix. [3 marks]

Substitute $\cos 60^\circ = 1/2$ and $\sin 60^\circ = \sqrt{3}/2$.

Answer: $R=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$.

(b) Find the image of $(2,0)$ and give the inverse transformation. [3 marks]

Twice the first column gives the image.

Use the opposite angle to undo the rotation.

Answer: $(1,\sqrt{3})$; rotate 60° clockwise.

5. Let P_x be projection onto the x-axis and R a 90° counterclockwise rotation. [6 marks]

(a) Find P_xR and apply it to $(3, 4)$. [3 marks]

Rotate to (-4,3), then remove the second component.

Answer: $P_xR = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$; image $(-4, 0)$.

(b) Find RP_x and apply it to the same point. Is either product invertible? [3 marks]

Project to (3,0), then rotate.

Both determinants are zero.

Answer: $RP_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; image $(0, 3)$. Neither is invertible.

6. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and solve $XA = \begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix}$. [6 marks]

(a) Find X. [3 marks]

Multiply on the right by $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

Answer: $X = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.

(b) Calculate AX and explain why it need not equal XA. [3 marks]

Calculate the product in the new order.

Matrix multiplication is not commutative.

Answer: $AX = \begin{pmatrix} 0 & 7 \\ -1 & 3 \end{pmatrix}$.

7. A map has matrix $M = \begin{pmatrix} k & 1 \\ 2 & k \end{pmatrix}$. A region of area 3 becomes a region of area 21. [6 marks]

(a) Find every possible real k. [4 marks]

$|\det M| = 7$, so $|k^2 - 2| = 7$.

The equations give k squared equal to 9 or -5; only 9 is possible over the reals.

Answer: $k = 3$ or $k = -3$.

(b) For each permitted k, state whether orientation is preserved. [2 marks]

Each permitted value gives determinant 7.

Answer: It is preserved for both.

8. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. [6 marks]

(a) Find every input mapped to $(5, 10)$. [3 marks]

The first image component gives x+2y=5.

The second gives the same equation multiplied by 2.

Answer: $(5 - 2t, t)$ for real t.

(b) Explain why no input maps to $(5, 11)$. [3 marks]

If X=5, the second component must be 10.

Answer: All images satisfy Y=2X.

9. The unit square is transformed by $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. [7 marks]

(a) Give its four image vertices in cyclic order and find its area. [4 marks]

Map the vertices (0,0),(1,0),(1,1),(0,1).

The determinant is 6.

Answer: $(0, 0), (2, 1), (2, 4), (0, 3)$; area 6.

(b) Find the acute angle between the two edges meeting at the origin. [3 marks]

Use edge vectors (2,1) and (0,3), with dot product 3 and lengths $\sqrt{5}, 3$.

Answer: $\theta = \cos^{-1}(1/\sqrt{5}) \approx 63.4^\circ$.

10. A rotation through 90° counterclockwise is about the centre $c = (1, -2)$. [6 marks]

(a) Find the image of $p = (4, -1)$. [3 marks]

Relative to c the vector is (3,1).

Rotate it to (-1,3), then add c.

Answer: $(0, 1)$.

(b) Find the input whose image is $(2, 4)$. [3 marks]

The image relative to c is (1,6).

Rotate clockwise to (6,-1), then add c.

Answer: $(7, -3)$.