Matrices and transformations - Practice test 4

10 questions | 63 marks | 70 minutes

Reasoning and proof: Explain matrix claims with proofs, counterexamples and careful dimension checks.

Equivalent correct methods earn the corresponding marks.

1. Let $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. [6 marks]

(a) Apply J to (1,0) and explain why J is not the identity. [3 marks]

The identity must leave (1,0) unchanged.

Answer: Image $(1, 1)$.

(b) Find every vector left fixed by J. [3 marks]

Equality (x+y,x+y)=(x,y) implies y=0 from the first component and x=0 from the second.

Answer: Only the zero vector.

2. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. [6 marks]

(a) Calculate AB and explain what this disproves. [3 marks]

Neither factor is zero.

Thus a zero matrix product need not have a zero factor.

Answer: $AB = 0$.

(b) Prove that if C is invertible and CD=0, then D=0. [3 marks]

Multiply on the left by C inverse.

Use the inverse identity.

Answer: $D = C^{-1}CD = C^{-1}0 = 0$.

3. Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $F=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. [7 marks]

(a) Find the conditions on the entries for MF=FM. [4 marks]

MF has entries (a,-b;c,-d).

FM has entries (a,b;-c,-d); equality forces b and c to vanish.

Answer: $b = 0$ and $c = 0$.

(b) If M also commutes with $R=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, what further condition is needed? [3 marks]

For diagonal M, compare the off-diagonal entries of MR and RM.

Answer: $a = d$, so M is a scalar identity.

4. The matrix $P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects onto the x-axis. [6 marks]

(a) Show that applying P twice has the same effect as once. [3 marks]

Direct multiplication gives the same diagonal entries 1 and 0.

Answer: $P^2 = P$.

(b) Describe the image of the entire plane and give two different inputs with the same non-zero image. [3 marks]

The coordinate rule is (x,y) to (x,0).

Answer: Image: x-axis. For example (2,1) and (2,-3) both map to (2,0).

5. Two linear maps have $\det A = -2$ and $\det B = 3$. [6 marks]

(a) A region has area 5. Find its area after A then B, and state the orientation change. [3 marks]

The combined determinant is 3(-2)=-6.

The absolute area multiplier is 6.

Answer: Area 30; orientation reverses.

(b) Find the area multiplier of the inverse combined map. [3 marks]

The combined map is invertible.

The inverse determinant is the reciprocal.

Answer: $\frac{1}{6}$.

6. Two matrices A and B are claimed to have AB of order 2 by 4 and BA of order 3 by 3. [6 marks]

(a) Show that no pair of matrices can meet both statements. [4 marks]

AB being 2 by 4 means A has 2 rows and B has 4 columns.

BA would therefore require 4 inner entries to match 2, which is impossible.

Answer: The dimensions are inconsistent.

(b) Give compatible orders for A and B for which AB is 2 by 2 and BA is 3 by 3. [2 marks]

The inner dimensions match in both products.

Answer: A is 2 by 3; B is 3 by 2.

7. Let $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. [6 marks]

(a) Find every point fixed by S and identify the transformation. [3 marks]

Solve (y,x)=(x,y).

Answer: Fixed points have $y = x$; S is reflection in that line.

(b) Show that S is its own inverse and interpret the determinant. [3 marks]

Swapping coordinates twice restores them.

The determinant shows unchanged area and reversed orientation.

Answer: $S^2 = I$, determinant -1.

8. Let $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. [6 marks]

(a) Describe the output line and all inputs mapped to zero. [3 marks]

The second output is -2 times the first.

For zero output require 2x-y=0.

Answer: Output line $Y = -2X$; zero inputs $(t, 2t)$.

(b) Find all inputs mapped to $(3, -6)$. [3 marks]

The first output equation is 2x-y=3.

The second is dependent.

Answer: $(t, 2t - 3)$ for real t.

9. Let R be a 90° counterclockwise rotation and t=(a,b) a translation vector. [7 marks]

(a) Express the two rules: translate by t then rotate; rotate then translate. [3 marks]

Distribute R over the vector sum.

Answer: $R(p+t) = Rp + Rt$; $Rp + t$.

(b) For which t are the two rules equal for every input p? [4 marks]

Equality requires Rt=t.

Thus (-b,a)=(a,b), giving a=b and a=-b, hence both zero.

Answer: Only $t = (0, 0)$.

10. Let $M = \begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$. [7 marks]

(a) Show that M squared is I for every real k. [3 marks]

The upper-right entry is k-k=0.

The diagonal entries are both 1.

Answer: $M^2 = I$.

(b) For which k is M an ordinary orthogonal reflection in the x-axis? Justify. [4 marks]

The map fixes the x-axis.

The image of the second unit vector is (k,-1), whose length is 1 only when k=0.

Answer: Only $k = 0$.