Complex numbers - reasoning test

10 questions | 56 marks | 75 minutes

Use component equations, geometric constraints and short proofs to justify complex-number results.

Equivalent correct methods earn the corresponding marks.

1. Let $z = -2 - 3i$. [5 marks]

(a) Find $-\overline{z}$ and $\overline{-z}$. [3 marks]

$\overline{z} = -2 + 3i$, so $-\overline{z} = 2 - 3i$.

$-z = 2 + 3i$, whose conjugate is $2 - 3i$.

Answer: Both are $2 - 3i$.

(b) Explain why the equality holds for every complex z. [2 marks]

For $z = a + bi$, both expressions simplify to $-a + bi$.

Answer: Conjugation respects multiplication by the real number -1.

2. Consider $(2 + i)z = 5$. [5 marks]

(a) Solve for z. [2 marks]

$z = 5(2 - i)/5$.

Answer: $z = 2 - i$.

(b) Use moduli to check the magnitude of your result. [3 marks]

$|2 + i||z| = 5$ gives $|z| = 5/\sqrt{5} = \sqrt{5}$.

The computed root $2 - i$ also has modulus $\sqrt{5}$.

Answer: $|z| = \sqrt{5}$.

3. Use the cycle of powers of i. [5 marks]

(a) Evaluate $i^{25} - i^{14} + i^{8}$. [3 marks]

The three powers are $i, -1, 1$.

Subtracting -1 contributes $+1$.

Answer: $2 + i$.

(b) Find i^{-1} and explain why it is not i. [2 marks]

$i(-i) = -i^2 = 1$.

But $i(i) = -1$, so i is not its own reciprocal.

Answer: $-i$.

4. A complex number satisfies $z + 2\overline{z} = 9 - 4i$. [6 marks]

(a) Find z by setting $z = a + bi$. [3 marks]

$z + 2\overline{z} = 3a - bi$.

$3a = 9$ and $-b = -4$.

Answer: $z = 3 + 4i$.

(b) Find $z/\overline{z}$ in Cartesian form. [3 marks]

$z/\overline{z} = (3 + 4i)^2/25$.

The numerator is $-7 + 24i$.

Answer: $\frac{-7 + 24i}{25}$.

5. A monic real quadratic $z^2 + pz + q = 0$ has a root $2i$. [5 marks]

(a) Find p and q. [3 marks]

The other root is $-2i$.

The factor product is $(z - 2i)(z + 2i) = z^2 + 4$.

Answer: $p = 0$, $q = 4$.

(b) For a nonzero real b, state the monic real quadratic with roots $\pm bi$. [2 marks]

Expand $(z - bi)(z + bi) = z^2 + b^2$.

Answer: $z^2 + b^2 = 0$.

6. Let $p(z) = z^2 + az + b$ with real a, b, and suppose $p(1 + 2i) = 0$. [6 marks]

(a) Use substitution and component equations to find a, b. [4 marks]

$(1 + 2i)^2 = -3 + 4i$.

Imaginary parts: $4 + 2a = 0$, so $a = -2$.

Real parts: $-3 + a + b = 0$, so $b = 5$.

Answer: $a = -2$, $b = 5$.

(b) Verify that the conjugate is also a root. [2 marks]

$(1 - 2i)^2 = -3 - 4i$.

Add $-2(1 - 2i) + 5$ to obtain zero.

Answer: $p(1 - 2i) = 0$.

7. A quadratic $q(z)$ has roots $-1 \pm i$ and satisfies $q(0) = 6$. [6 marks]

(a) Find its leading coefficient and expanded expression. [4 marks]

The monic factor product is $(z + 1)^2 + 1 = z^2 + 2z + 2$.

If $q = k(z^2 + 2z + 2)$, then $q(0) = 2k = 6$, giving $k = 3$.

Answer: Leading coefficient 3; $q(z) = 3z^2 + 6z + 6$.

(b) Explain why the roots alone did not determine the leading coefficient. [2 marks]

Multiplying a polynomial by a nonzero constant does not change where it equals zero.

Answer: Any nonzero scalar multiple has the same zero set.

8. Point Z represents $z = a + 2i$, with real a. It is equally far from 1 and $-3 + 4i$. [6 marks]

(a) Form an equation in a using squared distances. [3 marks]

Both vertical differences have magnitude 2.

Use the distance formula on each side.

Answer: $(a-1)^2 + 4 = (a+3)^2 + 4$.

(b) Find z and the common distance. [3 marks]

Expanding gives $-2a + 1 = 6a + 9$, hence $a = -1$.

The squared distance is $(-2)^2 + 2^2 = 8$.

Answer: $z = -1 + 2i$; common distance $2\sqrt{2}$.

9. Take $z = a + bi$ and $w = c + di$, with real components. [6 marks]

(a) Express the real and imaginary components of zw. [2 marks]

Expand the product and replace i^2 by -1.

Answer: $zw = (ac - bd) + (ad + bc)i$.

(b) Prove $|zw| = |z||w|$. [4 marks]

$|zw|^2 = (ac - bd)^2 + (ad + bc)^2$.

Expansion cancels the cross terms and gives $(a^2 + b^2)(c^2 + d^2)$.

This is $|z|^2|w|^2$; take non-negative square roots.

Answer: The two non-negative expressions have equal squares.

10. A complex number z has real part 2 and is at distance $\sqrt{10}$ from the point 1. [6 marks]

(a) Find all possible values of z. [3 marks]

Write $z = 2 + bi$.

$|z - 1|^2 = 1 + b^2 = 10$, giving $b = \pm 3$.

Answer: $z = 2 \pm 3i$.

(b) Find the monic real quadratic whose roots are these two values. [3 marks]

The roots have sum 4 and product $4 + 9 = 13$.

Answer: $z^2 - 4z + 13 = 0$.