Complex numbers - extended mixed test

Name: _____ Date: _____

10 questions | 60 marks | 75 minutes

Combine exact calculation, constraints, geometric reasoning and factorisation. Later questions require several linked ideas.

No calculator needed.

1. Begin with exact Cartesian calculations. [5 marks]

(a) Simplify $i^{42} + 2i^{19}$. [2 marks]

(b) Calculate $(2 - i)^2 - (1 + 3i)$. [3 marks]

2. Let $u = \frac{1+2i}{2+i}$. [6 marks]

(a) Find u in Cartesian form. [3 marks]

(b) Show that $u^{-1} = \overline{u}$. [3 marks]

3. The polynomial $p(z) = z^2 - (1+3i)z - 4 + 3i$ has root $2 + i$. [6 marks]

(a) Find its other root and factorise p. [4 marks]

(b) Explain why the two roots need not be conjugates. [2 marks]

4. The conjugate roots of a monic real quadratic have midpoint 3 on the Argand diagram and are 8 units apart. [6 marks]

(a) Find the roots. [3 marks]

(b) Find the quadratic and the modulus of each root. [3 marks]

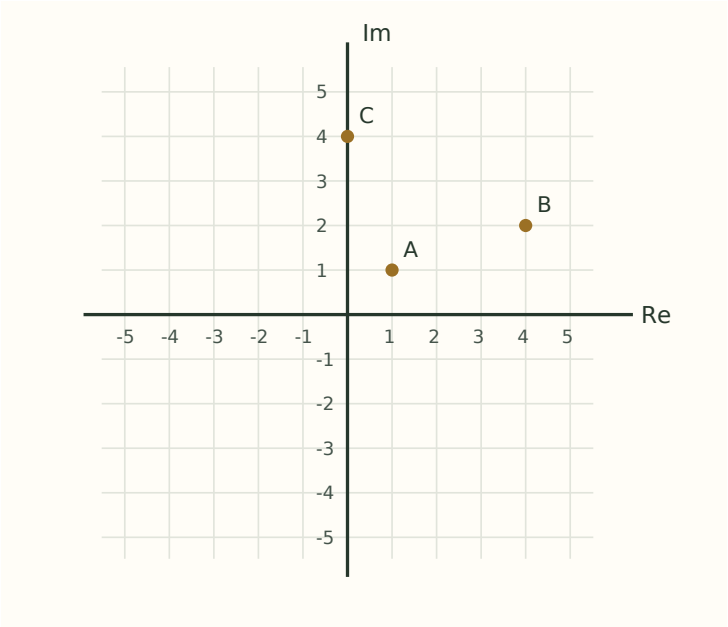
5. For real t, define $w = \frac{t+i}{t-i}$. [7 marks]

(a) Show that $w = \frac{t^2-1}{t^2+1} + \frac{2t}{t^2+1}i$. [3 marks]

(b) Show that $|w| = 1$. [2 marks]

(c) Find t when w is pure imaginary. [2 marks]

6. A triangle has complex vertices $A = 1 + i$, $B = 4 + 2i$ and $C = 4i$. [6 marks]



(a) Show it is a right-angled isosceles triangle. [4 marks]

(b) Find its area. [2 marks]

7. Consider $p(z) = (z^2 - 2z + 5)^2$. [5 marks]

(a) Factor $p(z)$ completely into complex linear factors. [3 marks]

(b) List the distinct roots and their multiplicities. [2 marks]

8. A supplied transmission model sends z to $w = (1 - 2i)z$. An input is $z = 3 + 4i$. [6 marks]

(a) Find the output and its magnitude. [3 marks]

(b) Find the factor by which this model scales every nonzero input magnitude. [3 marks]

9. Let $z = a + bi$ with real a, b. [7 marks]

(a) Show that z^2 has imaginary part $2ab$. [2 marks]

(b) Classify all z for which z^2 is positive real. [3 marks]

(c) Classify all z for which z^2 is negative real. [2 marks]

10. For $z = a + bi$ with $a \neq 0$, define $R = \frac{z - \bar{z}}{z + \bar{z}}$. [6 marks]

(a) Show that $R = \frac{b}{a}i$. [2 marks]

(b) Find all z satisfying $R = 2i$ and $|z| = \sqrt{5}$. [4 marks]