Proofs of summation formulas

6 questions | 36 marks

Equivalent correct methods earn the corresponding marks.

1. Prove $1 + 2 + \cdots + n = n(n+1)/2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to arbitrary integer $k \geq 1$ is $k(k+1)/2$.

Answer: At $n = 1$, both sides are 1. Assume the sum to arbitrary integer $k \geq 1$ is $k(k+1)/2$.

(b) Prove the next case and conclude. [4 marks]

Add the next term $k + 1$.

The next sum is $k(k+1)/2 + (k+1) = (k+1)(k+2)/2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

2. Prove $1 + 3 + \cdots + (2n-1) = n^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $1 = 1^2$.

Assume the first k odd numbers sum to k^2.

Answer: At $n = 1$, $1 = 1^2$. Assume the first k odd numbers sum to k^2.

(b) Prove the next case and conclude. [4 marks]

Add the next odd term $2k + 1$.

The result is $k^2 + 2k + 1 = (k+1)^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $1 = 1(2)(3)/6$.

Assume the formula holds at arbitrary $n = k$.

Answer: At $n = 1$, $1 = 1(2)(3)/6$. Assume the formula holds at arbitrary $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^2$ and factor $(k+1)/6$.

The remaining factor is $k(2k+1) + 6(k+1) = 2k^2 + 7k + 6$.

Factor it as $(k+2)(2k+3)$, giving the required formula at $k+1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\sum_{r=1}^{n} r^3 = [n(n+1)/2]^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $k^2(k+1)^2/4$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $k^2(k+1)^2/4$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^3$ and factor $(k+1)^2/4$.

The remaining factor is $k^2 + 4k + 4 = (k+2)^2$.

The result is $[(k+1)(k+2)/2]^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $\sum_{r=1}^{n} 1/[r(r+1)] = n/(n+1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1/2.

Assume the sum to k is $k/(k+1)$.

Answer: At $n = 1$, both sides are 1/2. Assume the sum to k is $k/(k+1)$.

(b) Prove the next case and conclude. [4 marks]

Add $1/[(k+1)(k+2)]$.

The numerator over the common denominator is $k(k+2)+1 = (k+1)^2$.

Cancellation gives $(k+1)/(k+2)$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)/3$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $2 = 1(2)(3)/3$.

Assume the formula holds at $n = k$.

Answer: At $n = 1$, $2 = 1(2)(3)/3$. Assume the formula holds at $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)(k+2)$.

Factoring gives $(k+1)(k+2)(k+3)/3$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.