Divisibility proofs

6 questions | 36 marks

Equivalent correct methods earn the corresponding marks.

1. Prove $5^n - 1$ is divisible by 4 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 4.

Assume $5^k - 1 = 4q$ for an integer q.

Answer: $n = 1$ gives 4. Assume $5^k - 1 = 4q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $5^{k+1} - 1 = 5(5^k - 1) + 4$.

This is $4(5q + 1)$, an integer multiple of 4.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

2. Prove $5^n - 2^n$ is divisible by 3 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 3.

Assume $5^k - 2^k = 3q$ for an integer q.

Answer: $n = 1$ gives 3. Assume $5^k - 2^k = 3q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $5^{k+1} - 2^{k+1} = 5(5^k - 2^k) + (5 - 2)2^k$.

This is $3(5q + 2^k)$, an integer multiple of 3.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $8^n - 3^n$ is divisible by 5 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 5.

Assume $8^k - 3^k = 5q$ for an integer q.

Answer: $n = 1$ gives 5. Assume $8^k - 3^k = 5q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $8^{k+1} - 3^{k+1} = 8(8^k - 3^k) + (8 - 3)3^k$.

This is $5(8q + 3^k)$, an integer multiple of 5.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is $0 = 6(0)$.

Assume $k^3 - k = 6q$ for an integer q.

Answer: At $n = 1$, the value is $0 = 6(0)$. Assume $k^3 - k = 6q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$(k + 1)^3 - (k + 1) = (k^3 - k) + 3k(k + 1)$.

One of consecutive integers $k, k + 1$ is even, so $k(k + 1) = 2t$ for an integer t.

The next value is $6(q + t)$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is 3.

Assume $k^3 + 2k = 3q$ for an integer q.

Answer: At $n = 1$, the value is 3. Assume $k^3 + 2k = 3q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$(k+1)^3 + 2(k+1) = (k^3 + 2k) + 3(k^2 + k + 1)$.

This is $3(q + k^2 + k + 1)$, an integer multiple of 3.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $7^n - (-1)^n$ is divisible by 8 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is 8.

Assume $7^k - (-1)^k = 8q$ for an integer q.

Answer: At $n = 1$, the value is 8. Assume $7^k - (-1)^k = 8q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$7^{k+1} - (-1)^{k+1} = 7[7^k - (-1)^k] + 8(-1)^k$.

This is $8[7q + (-1)^k]$, divisible by 8.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.