Identities involving powers and integers

6 questions | 36 marks

Equivalent correct methods earn the corresponding marks.

1. Prove $1 + 2 + 2^2 + \cdots + 2^{n-1} = (2^n - 1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $1 = 1$.

Assume the sum to k is $(2^k - 1)$.

Answer: $n = 1$ gives $1 = 1$. Assume the sum to k is $(2^k - 1)$.

(b) Prove the next case and conclude. [4 marks]

Add 2^k.

The new total is $2^k - 1 + 2^k = 2^{k+1} - 1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

2. Prove $1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n - 1)/2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $(3 - 1)/2 = 1$.

Assume the sum to k is $(3^k - 1)/2$.

Answer: $n = 1$ gives $(3 - 1)/2 = 1$. Assume the sum to k is $(3^k - 1)/2$.

(b) Prove the next case and conclude. [4 marks]

Add 3^k.

The common numerator is $3^k - 1 + 2(3^k) = 3^{k+1} - 1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} r2^{r-1} = (n-1)2^n + 1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $(k-1)2^k + 1$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $(k-1)2^k + 1$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)2^k$.

The power terms combine to $2k2^k = k2^{k+1}$, giving the required formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\prod_{r=1}^{n} (r+1)/r = n + 1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is 2.

Assume the product to k is $k + 1$.

Answer: At $n = 1$, the product is 2. Assume the product to k is $k + 1$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $(k+2)/(k+1)$.

The product becomes $k + 2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $\prod_{r=1}^{n}(2r-1)/(2r+1) = 1/(2n+1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is $1/3$.

Assume the product to k is $1/(2k+1)$.

Answer: At $n = 1$, the product is $1/3$. Assume the product to k is $1/(2k+1)$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $(2k+1)/(2k+3)$.

The result is $1/(2k+3) = 1/[2(k+1)+1]$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $\sum_{r=1}^{n}(2r-1)^2 = n(4n^2-1)/3$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides equal 1.

Assume the formula holds at $n = k$.

Answer: At $n = 1$, both sides equal 1. Assume the formula holds at $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(2k+1)^2$.

The numerator becomes $4k^3 + 12k^2 + 11k + 3 = (k+1)(4k^2+8k+3)$.

This equals $(k+1)[4(k+1)^2 - 1]$, as required.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.