Mathematical induction - Practice test 1

10 questions | 58 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A paving design starts with four blocks. Each successive stage contains exactly three more blocks than the preceding stage. [5 marks]

(a) Find a rule for the number in stage n. [2 marks]

The construction gives an arithmetic pattern with first value 4 and difference 3.

Answer: $3n + 1$.

(b) Suggest the cumulative total through stage n, and explain how it could be proved. [3 marks]

Sum the stage rule from 1 to n.

Check n=1, assume the kth total, then add the (k+1)th stage.

Answer: $T_n = 3n(n + 1)/2 + n$.

2. Prove $1 + 2 + \cdots + n = n(n + 1)/2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to arbitrary integer $k \geq 1$ is $k(k + 1)/2$.

Answer: At $n = 1$, both sides are 1. Assume the sum to arbitrary integer $k \geq 1$ is $k(k + 1)/2$.

(b) Prove the next case and conclude. [4 marks]

Add the next term $k + 1$.

The next sum is $k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $1 + 3 + \cdots + (2n - 1) = n^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $1 = 1^2$.

Assume the first k odd numbers sum to k^2.

Answer: At $n = 1$, $1 = 1^2$. Assume the first k odd numbers sum to k^2.

(b) Prove the next case and conclude. [4 marks]

Add the next odd term $2k + 1$.

The result is $k^2 + 2k + 1 = (k + 1)^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\sum_{r=1}^{n} r^2 = n(n + 1)(2n + 1)/6$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $1 = 1(2)(3)/6$.

Assume the formula holds at arbitrary $n = k$.

Answer: At $n = 1$, $1 = 1(2)(3)/6$. Assume the formula holds at arbitrary $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k + 1)^2$ and factor $(k + 1)/6$.

The remaining factor is $k(2k + 1) + 6(k + 1) = 2k^2 + 7k + 6$.

Factor it as $(k + 2)(2k + 3)$, giving the required formula at $k + 1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n - 1)/2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $(3 - 1)/2 = 1$.

Assume the sum to k is $(3^k - 1)/2$.

Answer: $n = 1$ gives $(3 - 1)/2 = 1$. Assume the sum to k is $(3^k - 1)/2$.

(b) Prove the next case and conclude. [4 marks]

Add 3^k.

The common numerator is $3^k - 1 + 2(3^k) = 3^{k+1} - 1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $5^n - 2^n$ is divisible by 3 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 3.

Assume $5^k - 2^k = 3q$ for an integer q.

Answer: $n = 1$ gives 3. Assume $5^k - 2^k = 3q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $5^{k+1} - 2^{k+1} = 5(5^k - 2^k) + (5 - 2)2^k$.

This is $3(5q + 2^k)$, an integer multiple of 3.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

7. Prove $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is $0 = 6(0)$.

Assume $k^3 - k = 6q$ for an integer q.

Answer: At $n = 1$, the value is $0 = 6(0)$. Assume $k^3 - k = 6q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$(k+1)^3 - (k+1) = (k^3 - k) + 3k(k+1)$.

One of consecutive integers $k, k+1$ is even, so $k(k+1) = 2t$ for an integer t.

The next value is $6(q+t)$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

8. Prove $\sum_{r=1}^{n} 1/[r(r+1)] = n/(n+1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1/2.

Assume the sum to k is $k/(k+1)$.

Answer: At $n = 1$, both sides are 1/2. Assume the sum to k is $k/(k+1)$.

(b) Prove the next case and conclude. [4 marks]

Add $1/[(k+1)(k+2)]$.

The numerator over the common denominator is $k(k+2) + 1 = (k+1)^2$.

Cancellation gives $(k+1)/(k+2)$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

9. Prove $\prod_{r=1}^{n}(r+1)/r = n+1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is 2.

Assume the product to k is $k+1$.

Answer: At $n = 1$, the product is 2. Assume the product to k is $k+1$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $(k+2)/(k+1)$.

The product becomes $k+2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

10. A student assumes the result at k+1 and rearranges it to the hypothesis at k. Explain the logical gap. [5 marks]

(a) Identify the issue or give the required counterexample, with reasoning. [3 marks]

Derive P(k+1) from P(k).

If reversing algebra, explicitly justify that each step is reversible.

Answer: The direction written is the reverse of the required implication.

(b) Describe a valid repair of the proof direction. [2 marks]

The required direction is P(k) implies P(k+1).

Answer: Start from the kth hypothesis and derive the successor, justifying every reversal if working backwards.