Mathematical induction - Practice test 2

10 questions | 57 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A student wants to prove a formula for every integer n at least 2. [5 marks]

(a) State the required starting case and explain the role of an arbitrary k. [2 marks]

The implication must work at every allowed stage, not one fixed integer.

Answer: Check P(2), then use arbitrary k at least 2.

(b) A proposed implication moves from k to k+2. Give enough starting cases and explain why. [3 marks]

A step of two cannot change parity.

Answer: P(2) and P(3), one for each parity chain.

2. Prove $\sum_{r=1}^{n}(3r-1)=3n(n+1)/2-n$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives 2 on both sides.

Assume the formula holds at $n=k$.

Answer: $n=1$ gives 2 on both sides. Assume the formula holds at $n=k$.

(b) Prove the next case and conclude. [4 marks]

Add $3(k+1)-1$.

Grouping gives $3(k+1)(k+2)/2-(k+1)$, the required next formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} r^3 = [n(n+1)/2]^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $k^2(k+1)^2/4$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $k^2(k+1)^2/4$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^3$ and factor $(k+1)^2/4$.

The remaining factor is $k^2 + 4k + 4 = (k+2)^2$.

The result is $[(k+1)(k+2)/2]^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $1 + 2 + 2^2 + \cdots + 2^{n-1} = (2^n - 1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $1 = 1$.

Assume the sum to k is $(2^k - 1)$.

Answer: $n = 1$ gives $1 = 1$. Assume the sum to k is $(2^k - 1)$.

(b) Prove the next case and conclude. [4 marks]

Add 2^k.

The new total is $2^k - 1 + 2^k = 2^{k+1} - 1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $8^n - 3^n$ is divisible by 5 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 5.

Assume $8^k - 3^k = 5q$ for an integer q.

Answer: $n = 1$ gives 5. Assume $8^k - 3^k = 5q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $8^{k+1} - 3^{k+1} = 8(8^k - 3^k) + (8 - 3)3^k$.

This is $5(8q + 3^k)$, an integer multiple of 5.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is 3.

Assume $k^3 + 2k = 3q$ for an integer q.

Answer: At $n = 1$, the value is 3. Assume $k^3 + 2k = 3q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$(k + 1)^3 + 2(k + 1) = (k^3 + 2k) + 3(k^2 + k + 1)$.

This is $3(q + k^2 + k + 1)$, an integer multiple of 3.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

7. Prove $\sum_{r=1}^{n} r2^{r-1} = (n-1)2^n + 1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $(k-1)2^k + 1$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $(k-1)2^k + 1$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)2^k$.

The power terms combine to $2k2^k = k2^{k+1}$, giving the required formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

8. Prove $\prod_{r=1}^{n} (2r-1)/(2r+1) = 1/(2n+1)$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is $1/3$.

Assume the product to k is $1/(2k+1)$.

Answer: At $n = 1$, the product is $1/3$. Assume the product to k is $1/(2k+1)$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $(2k+1)/(2k+3)$.

The result is $1/(2k+3) = 1/[2(k+1)+1]$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

9. Prove $\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)/3$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $2 = 1(2)(3)/3$.

Assume the formula holds at $n = k$.

Answer: At $n = 1$, $2 = 1(2)(3)/3$. Assume the formula holds at $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)(k+2)$.

Factoring gives $(k+1)(k+2)(k+3)/3$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

10. Why do correct checks for the first one hundred positive integers not constitute induction? [4 marks]

(a) Identify the issue or give the required counterexample, with reasoning. [2 marks]

An untested case could still fail.

A starting case and a general successor implication are both required.

Answer: There is no implication proved for an arbitrary integer.

(b) Which induction obligation has already been met, and which remains? [2 marks]

The first hundred checked cases include n=1.

They do not establish P(k) implies P(k+1) for arbitrary positive k.

Answer: The base case n=1 has been checked. A successor implication for every arbitrary positive integer k remains to be proved.