Mathematical induction - Practice test 4

10 questions | 58 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A proposed identity is $\sum_{r=1}^{n}(2r+1)=n^2$. [5 marks]

(a) Test the first case and decide whether induction can prove this identity. [2 marks]

A false allowed case prevents a valid proof of the claim.

Answer: At n=1, the sum is 3 but the proposed value is 1, so the statement is false.

(b) Find and justify a corrected formula. [3 marks]

Summing $2r+1$ gives $2[n(n+1)/2]+n=n(n+2)$.

Answer: $n(n+2)$.

2. Prove $\sum_{r=1}^{n}(5r-2)=5n(n+1)/2-2n$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives 3 on both sides.

Assume the formula holds at $n=k$.

Answer: $n=1$ gives 3 on both sides. Assume the formula holds at $n=k$.

(b) Prove the next case and conclude. [4 marks]

Add $5(k+1)-2$.

Grouping gives $5(k+1)(k+2)/2-2(k+1)$, the required next formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} 1/[r(r+2)] = n(3n+5)/[4(n+1)(n+2)]$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are $1/3$.

Assume the stated expression holds for the sum to k.

Answer: At $n = 1$, both sides are $1/3$. Assume the stated expression holds for the sum to k.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $1/[(k+1)(k+3)]$.

Over $4(k+1)(k+2)(k+3)$, the numerator is $k(3k+5)(k+3) + 4(k+2)$.

Factor it as $(k+1)^2(3k+8)$ and cancel one factor $k+1$.

The result is $(k+1)(3k+8)/[4(k+2)(k+3)]$, exactly the next formula.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove $\sum_{r=1}^{n} r/(r+1)! = 1 - 1/(n+1)!$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are $1/2$.

Assume the sum to k is $1 - 1/(k+1)!$.

Answer: At $n = 1$, both sides are $1/2$. Assume the sum to k is $1 - 1/(k+1)!$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(k+1)/(k+2)!$.

Since $1/(k+1)! = (k+2)/(k+2)!$, the difference becomes $1 - 1/(k+2)!$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $(1-x)(1+x+\cdots+x^n)=1-x^{n+1}$ for integers $n \geq 1$ and every real x. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1$, $(1-x)(1+x)=1-x^2$.

Assume the identity through the power x^k, for an arbitrary real x. Keep x fixed through the induction step.

Answer: At $n=1$, $(1-x)(1+x)=1-x^2$. Assume the identity through the power x^k, for an arbitrary real x. Keep x fixed through the induction step.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(1-x)x^{k+1}$ to the assumed left side.

The right side becomes $1-x^{k+1}+(1-x)x^{k+1}=1-x^{k+2}$.

No division by $1-x$ was used, so $x=1$ is included.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

6. Set $A_n=n^4+2n^3-n^2-2n$ for positive integers n. [6 marks]

(a) Show $A_{k+1}-A_k=4k(k+1)(k+2)$ and explain why this difference is divisible by 24. [3 marks]

Expanding and subtracting gives $4k^3+12k^2+8k=4k(k+1)(k+2)$.

Among three consecutive integers there is a multiple of 3 and at least one even integer. Their product is a multiple of 6. Multiplication by 4 makes a multiple of 24.

Answer: $A_{k+1}-A_k=4k(k+1)(k+2)$, a multiple of 24.

(b) Use induction to prove that 24 divides An throughout the stated domain. [3 marks]

$A_1=0=24(0)$. Assume $A_k=24q$ for an integer q.

By part (a), $4k(k+1)(k+2)=24t$ for an integer t, so $A_{k+1}=24(q+t)$.

The base and implication establish the claim.

Answer: 24 divides A_n for every positive integer n.

7. A display has n sections. Section r contains $(2r-1)^3$ illuminated points. [6 marks]

(a) Write the first three section sizes and verify the total formula $T_n = n^2(2n^2-1)$ at n=1. [2 marks]

The section rule uses odd-number cubes. The total includes every section from 1 to n.

Answer: The sizes are 1, 27 and 125. At n=1, $T_1 = 1^2(2-1) = 1$.

(b) Prove the total formula by induction. [4 marks]

Assume $T_k = k^2(2k^2-1)$ for arbitrary positive integer k.

The new section has $(2k+1)^3$ points. Thus $T_{k+1} = 2k^4 - k^2 + (2k+1)^3$.

Expanding gives $2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k+1)^2[2(k+1)^2 - 1]$.

The checked base and successor implication prove the rule.

Answer: $T_n = n^2(2n^2-1)$ for all positive integers n.

8. A proposed formula has the form $\sum_{r=1}^{n} r(r+3) = n(n+1)(an+b)$, with constants a and b. [6 marks]

(a) Use n=1 and n=2 to determine a and b. Explain why this alone is not a proof. [2 marks]

At n=1, $4 = 2(a+b)$, so $a+b = 2$. At n=2, $14 = 6(2a+b)$, so $2a+b = 7/3$.

Subtraction gives a=1/3, then b=5/3.

Answer: $a = 1/3$, $b = 5/3$. Two checked cases do not establish every positive integer case.

(b) Prove the resulting formula for all positive integers n. [4 marks]

The first case is already checked. Assume the sum to k is $k(k+1)(k+5)/3$.

Add $(k+1)(k+4)$ and factor $(k+1)/3$.

The remaining polynomial is $k(k+5) + 3(k+4) = k^2 + 8k + 12 = (k+2)(k+6)$.

The next sum is $(k+1)(k+2)(k+6)/3$, as required. Conclude by induction.

Answer: $\sum_{r=1}^{n} r(r+3) = n(n+1)(n+5)/3$.

9. The first case has been checked for $\prod_{r=1}^{n}(2r-1)/(2r+1) = 1/(2n+1)$. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

Assume the product to k is $1/(2k+1)$.

Answer: Assume the product to k is $1/(2k+1)$.

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

Multiply by $(2k+1)/(2k+3)$.

The result is $1/(2k+3) = 1/[2(k+1)+1]$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

The chain starts at 1 and advances by one.

Answer: Every positive index is reached by successive one-step implications.

10. Refute the claim that $n^2 + n + 41$ is prime for every positive integer n. [5 marks]

(a) Identify the issue or give the required counterexample, with reasoning. [3 marks]

This allowed case is composite.

One counterexample refutes a universal claim.

Answer: $n = 41$ gives $41^2 + 41 + 41 = 41(43)$.

(b) Explain why one such counterexample is sufficient. [2 marks]

The allowed composite case already contradicts that assertion.

Answer: A universal assertion requires every allowed case to be true.