Mathematical induction - Practice test 5

10 questions | 57 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A triangular stack has n rows; row r contains exactly 2r tiles. [5 marks]

(a) Find the total tile count through row n. [2 marks]

It is twice $1 + 2 + \cdots + n$.

Answer: $n(n + 1)$.

(b) If the total formula holds through row k, write the next row count and obtain the next total. [3 marks]

$k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$.

Answer: The next row has $2(k + 1)$ tiles, giving $(k + 1)(k + 2)$ total.

2. Prove $\sum_{r=1}^{n} r(r + 1)(r + 2) = n(n + 1)(n + 2)(n + 3)/4$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides equal 6.

Assume the formula is true for the sum ending at k.

Answer: At $n = 1$, both sides equal 6. Assume the formula is true for the sum ending at k.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(k + 1)(k + 2)(k + 3)$.

Factor the common product to get $(k + 1)(k + 2)(k + 3)(k + 4)/4$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

3. Prove $\prod_{r=1}^{n}(r+2)/r=(n+1)(n+2)/2$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1$, both sides equal 3.

Assume the product to k is $(k+1)(k+2)/2$.

Answer: At $n=1$, both sides equal 3. Assume the product to k is $(k+1)(k+2)/2$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Multiply by the next factor $(k+3)/(k+1)$.

Cancel the nonzero $k+1$ to obtain $(k+2)(k+3)/2$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove 4^n+6n-1 is divisible by 9 for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1$, the value is 9.

Assume $A_k=4^k+6k-1=9q$ for an integer q.

Answer: At $n=1$, the value is 9. Assume $A_k=4^k+6k-1=9q$ for an integer q.

(b) Establish the successor and conclude the stated domain. [4 marks]

$A_{k+1}=4A_k+9(1-2k)$.

The right side is $9(4q+1-2k)$, an integer multiple of 9.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $11^n - 3^n$ is divisible by 8 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 8.

Assume $11^k - 3^k = 8q$ for an integer q.

Answer: $n = 1$ gives 8. Assume $11^k - 3^k = 8q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $11^{k+1} - 3^{k+1} = 11(11^k - 3^k) + (11 - 3)3^k$.

This is $8(11q + 3^k)$, an integer multiple of 8.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Consider the alternating sum $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$. [6 marks]

(a) Calculate S1, S2 and S3, then check the formula $S_n = (-1)^{n-1} n(n+1)/2$ at the first case. [2 marks]

The signs alternate: $1 - 4 + 9 = 6$.

Answer: $S_1 = 1$, $S_2 = -3$, $S_3 = 6$. The first-case formula is 1.

(b) Prove the formula by induction, taking care with the sign of the new term. [4 marks]

Assume $S_k = (-1)^{k-1} k(k+1)/2$. The new term is $(-1)^k (k+1)^2$.

Using $(-1)^{k-1} = -(-1)^k$ gives $S_{k+1} = (-1)^k (k+1)[-k/2 + (k+1)]$.

This simplifies to $(-1)^k (k+1)(k+2)/2$, the formula at k+1. The base and step prove the claim.

Answer: $S_n = (-1)^{n-1} n(n+1)/2$ for all positive integers n.

7. The first case has been checked for $\sum_{r=1}^{n}(2r-1)^2 = n(4n^2-1)/3$. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

Assume the formula holds at $n = k$.

Answer: Assume the formula holds at $n = k$.

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

Add $(2k+1)^2$.

The numerator becomes $4k^3 + 12k^2 + 11k + 3 = (k+1)(4k^2 + 8k + 3)$.

This equals $(k+1)[4(k+1)^2 - 1]$, as required.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

The chain starts at 1 and advances by one.

Answer: Every positive index is reached by successive one-step implications.

8. For positive integers n, let $D_n = \sum_{r=1}^{n} r/2^r$. [6 marks]

(a) Verify the proposed formula $D_n = 2 - (n+2)/2^n$ at n=1, and state an exact induction hypothesis. [2 marks]

The same arbitrary k must appear in the upper bound and both places on the right.

Answer: $D_1 = 1/2 = 2 - 3/2$. Assume $D_k = 2 - (k+2)/2^k$ for an arbitrary integer k at least 1.

(b) Prove the successor and use the formula to calculate D6 exactly. [4 marks]

Add $(k+1)/2^{k+1}$. The next sum is $2 - [2(k+2) - (k+1)]/2^{k+1} = 2 - (k+3)/2^{k+1}$.

This is the required successor, establishing the formula by induction.

$D_6 = 2 - 8/64 = 15/8$.

Answer: The formula holds for all positive integers n, and $D_6 = 15/8$.

9. For positive integers n, define the product of the first n odd numbers by $O_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. [6 marks]

(a) Explain the identity $(2n)! = 2^n n! O_n$ by grouping the even and odd factors. [2 marks]

There are n even factors $2, 4, \ldots, 2n$, whose product is $2^n(1 \cdot 2 \cdots n)$. The remaining factors are exactly those in O_n.

Answer: The even factors multiply to $2^n n!$, and the odd factors multiply to O_n.

(b) Independently prove $O_n = (2n)!/(2^n n!)$ by induction. [4 marks]

At n=1, $1 = 2!/(2 \cdot 1!)$. Assume the stated formula at k.

The next product is $(2k+1)(2k)!/(2^k k!)$.

The target is $(2k+2)!/[2^{k+1}(k+1)!]$. Expanding the factorials cancels $(2k+2)/[2(k+1)] = 1$ and leaves exactly the next product.

The first case and implication prove the formula.

Answer: The formula is true for every positive integer n.

10. A sum of odd numbers proof adds $2k-1$ to the assumed total k^2. Correct the next term. [4 marks]

(a) Identify the issue or give the required counterexample, with reasoning. [2 marks]

The assumed sum already contains $2k-1$.

The next odd number is $2(k+1)-1$.

Answer: It must add $2k+1$.

(b) Show the corrected successor total. [2 marks]

Expand the square on the right to verify the bridge.

Answer: $k^2 + (2k+1) = (k+1)^2$.