Complex Numbers - Paper 1

10 questions | 51 marks | 70 minutes

A balanced first paper on exact forms, arithmetic, powers, roots and simple geometry.

Equivalent correct methods earn the corresponding marks.

1. Let $z = -3 + 3\sqrt{3}\,i$. [4 marks]

(a) Find the modulus and principal argument. [2 marks]

$|z| = \sqrt{9 + 27} = 6$; the point is in quadrant II with reference angle $\pi/3$.

Answer: $|z| = 6$, $\operatorname{Arg} z = 2\pi/3$.

(b) Write all arguments of z. [2 marks]

Add any integer number of full turns to the principal argument.

Answer: $2\pi/3 + 2k\pi$, $k \in \mathbb{Z}$.

2. Let $u = 2 + i$ and $v = 1 - 3i$. [5 marks]

(a) Calculate uv in Cartesian form. [2 marks]

$(2 + i)(1 - 3i) = 2 - 6i + i - 3i^2 = 5 - 5i$.

Answer: $5 - 5i$.

(b) Calculate u/v in Cartesian form. [3 marks]

Multiply by $(1 + 3i)/(1 + 3i)$; numerator $-1 + 7i$, denominator 10.

Answer: $-\frac{1}{10} + \frac{7}{10}i$.

3. Consider $a = 4\text{cis}(\pi/6)$ and $b = 2\text{cis}(\pi/3)$. [5 marks]

(a) Find ab in Cartesian form. [2 marks]

The product is $8\text{cis}(\pi/2) = 8i$.

Answer: $8i$.

(b) Find a+b in Cartesian form. [3 marks]

$a = 2\sqrt{3} + 2i$ and $b = 1 + \sqrt{3}\,i$; add coordinates.

Answer: $(2\sqrt{3} + 1) + (2 + \sqrt{3})i$.

4. A navigation diagram represents a displacement by $z = 3 - 4i$. A turn multiplies the displacement by i. [5 marks]

(a) Find the new displacement. [2 marks]

$i(3 - 4i) = 4 + 3i$.

Answer: $4 + 3i$.

(b) Describe the turn and compare the two displacement lengths. [3 marks]

Multiplication by i adds $\pi/2$ to the argument and multiplies the modulus by 1.

Answer: An anticlockwise quarter-turn about the origin; both lengths are 5.

5. Use exact polar arithmetic. [5 marks]

(a) Calculate $(1 - \sqrt{3}\,i)^4$. [3 marks]

$1 - \sqrt{3}\,i = 2\text{cis}(-\pi/3)$. Its fourth power is $16\text{cis}(-4\pi/3) = 16\text{cis}(2\pi/3)$.

Answer: $-8 + 8\sqrt{3}\,i$.

(b) Calculate $(2\text{cis}(\pi/4))^{-2}$. [2 marks]

The result is $\frac{1}{4}\text{cis}(-\pi/2) = -i/4$.

Answer: $-i/4$.

6. Solve $z^3 = 27$. [6 marks]

(a) List all roots in polar form. [3 marks]

The common modulus is 3 and the angles are $0, 2\pi/3, 4\pi/3$.

Answer: $3\text{cis}(2k\pi/3)$, $k = 0, 1, 2$.

(b) Convert the two non-real roots to Cartesian form. [3 marks]

Use sine and cosine at $2\pi/3$ and $4\pi/3$.

Answer: $-\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$.

7. Let $\omega = \text{cis}(2\pi/3)$. [5 marks]

(a) Simplify $\omega^7 + \omega^5$. [2 marks]

Reduce powers modulo 3: $\omega^7 + \omega^5 = \omega + \omega^2 = -1$.

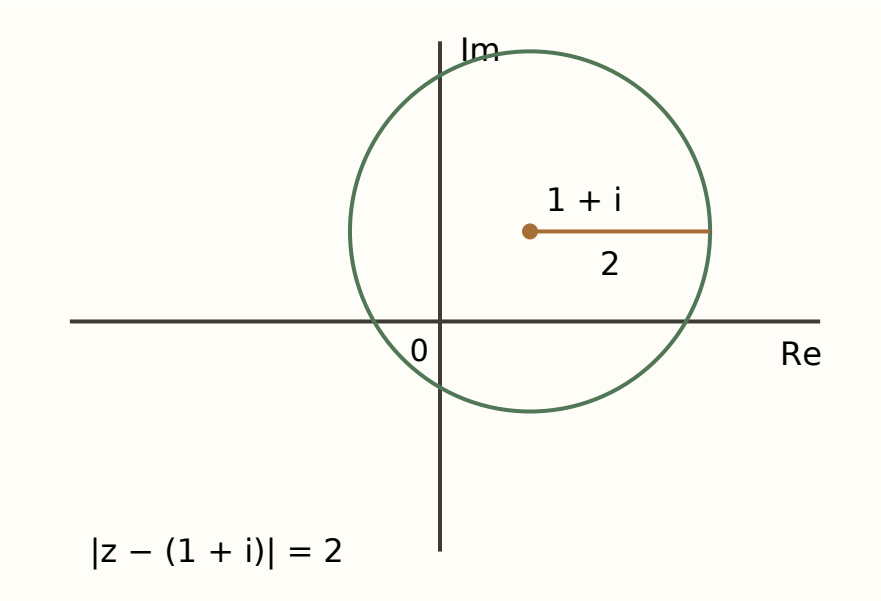
Answer: -1.

(b) Find $|1 - \omega|$. [3 marks]

$1 - \omega = 3/2 - (\sqrt{3}/2)i$; its squared modulus is $9/4 + 3/4 = 3$.

Answer: $\sqrt{3}$.

8. A locus is defined by $|z - (1 + i)| = 2$. [5 marks]



(a) State its centre and Cartesian equation. [2 marks]

The modulus measures the distance from $1 + i$.

Answer: Centre $(1, 1)$; $(x - 1)^2 + (y - 1)^2 = 4$.

(b) Find the points on this locus with real part 1. [3 marks]

Substitute $x = 1$: $(y - 1)^2 = 4$, so $y = 3$ or -1.

Answer: $z = 1 + 3i$ and $z = 1 - i$.

9. Consider $P(z) = z^4 + 10z^2 + 9$. [5 marks]

(a) Factorise P into real quadratic factors. [2 marks]

For $u = z^2$, $u^2 + 10u + 9 = (u + 1)(u + 9)$.

Answer: $(z^2 + 1)(z^2 + 9)$.

(b) Factorise P completely over the complex numbers. [3 marks]

The two quadratic factors have roots $\pm i$ and $\pm 3i$.

Answer: $(z - i)(z + i)(z - 3i)(z + 3i)$.

10. A monic polynomial of degree 3 has real coefficients and roots $2 + i$ and -1. [6 marks]

(a) State the third root, with a reason. [2 marks]

Real coefficients force the conjugate of the non-real root.

Answer: $2 - i$.

(b) Find the polynomial in expanded form. [4 marks]

The conjugate pair gives $z^2 - 4z + 5$. Multiply by $z + 1$ and collect terms.

Answer: $z^3 - 3z^2 + z + 5$.