Complex Numbers - Paper 3

10 questions | 56 marks | 70 minutes

Reasoning and proof: inspect claims, derive identities and connect algebra with geometry.

Equivalent correct methods earn the corresponding marks.

1. A student converts $-2+2i$ to polar form using $\tan^{-1}(-1)=-\pi/4$. [5 marks]

(a) Explain the quadrant error and give the correct principal-argument polar form. [3 marks]

The student's angle lies in quadrant IV, whereas the point has negative real and positive imaginary parts. Its reference angle is $\pi/4$.

Answer: $2\sqrt{2}\,\mathrm{cis}(3\pi/4)$.

(b) Give another argument outside the principal interval. [2 marks]

Add 2π to $3\pi/4$.

Answer: $11\pi/4$ is one answer.

2. Let $z=x+iy$, where x and y are real. [5 marks]

(a) Prove $z\overline{z}=|z|^2$. [3 marks]

$(x+iy)(x-iy)=x^2-(iy)^2=x^2+y^2$. The modulus squared has the same expression.

Answer: Both expressions equal x^2+y^2.

(b) Hence show that $1/z=\overline{z}$ when $|z|=1$. [2 marks]

The condition implies $z\neq 0$ and $z\overline{z}=1$. Divide by z.

Answer: $1/z=\overline{z}$.

3. Let $u = r\text{cis}\,\alpha$ and $v = s\text{cis}\,\beta$, where r and s are positive. [6 marks]

(a) Prove $|uv| = |u||v|$. [3 marks]

Polar multiplication gives $uv = rs\text{cis}(\alpha + \beta)$. Since $r, s > 0$, its modulus is $rs = |u||v|$.

Answer: The product modulus is rs.

(b) Is $\text{Arg}(uv) = \text{Arg}\,u + \text{Arg}\,v$ always literally true? Give a counterexample under $(-\pi, \pi]$. [3 marks]

Arguments add modulo 2π; the sum may need reducing to the principal interval.

Answer: No. For $u = v = -1$, the left side is 0 while the sum on the right is 2π.

4. Let $c = \cos\theta$ and $s = \sin\theta$. [6 marks]

(a) Expand $(c + is)^3$ into real and imaginary components. [3 marks]

Use the binomial expansion and $i^2 = -1$, $i^3 = -i$.

Answer: $(c^3 - 3cs^2) + i(3c^2s - s^3)$.

(b) Use de Moivre to prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3 marks]

Compare real parts: $\cos 3\theta = c^3 - 3cs^2$. Substitute $s^2 = 1 - c^2$ and simplify.

Answer: $\cos 3\theta = 4c^3 - 3c$.

5. A proposed solution of $z^3 = 1$ lists only $z = 1$, saying that the cube root of 1 is 1. [5 marks]

(a) Find the two omitted roots in Cartesian form. [3 marks]

The arguments $2\pi/3$ and $4\pi/3$ also triple to full turns.

Answer: $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

(b) Explain the difference between one principal root and solving the equation. [2 marks]

The extra arguments arise by adding full turns before dividing by 3.

Answer: A chosen principal cube root is a single value, but solving the equation over the complex numbers requires all three distinct solutions.

6. Let $\omega = \text{cis}(2\pi/3)$. [6 marks]

(a) Prove $1 + \omega + \omega^2 = 0$ without decimal approximations. [3 marks]

Use $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1 = 0$ and $\omega \neq 1$.

Answer: $1 + \omega + \omega^2 = 0$.

(b) Hence show that $(1 - \omega)(1 - \omega^2) = 3$. [3 marks]

Expand to $1 - (\omega + \omega^2) + \omega^3$. Substitute $\omega + \omega^2 = -1$ and $\omega^3 = 1$.

Answer: The product equals 3.

7. Two points A and B represent distinct complex numbers a and b. Their midpoint represents $m = (a + b)/2$. [6 marks]

(a) Show that $|m - a| = |m - b|$. [3 marks]

$m - a = (b - a)/2$ and $m - b = (a - b)/2$; opposite complex numbers have equal moduli.

Answer: Both distances are $|b - a|/2$.

(b) Does being equidistant from A and B force a point to be their midpoint? Give a concrete counterexample. [3 marks]

$|i + 1| = |i - 1| = \sqrt{2}$. All points on the perpendicular bisector are equidistant.

Answer: No. For $a = -1$, $b = 1$, the point $z = i$ is equidistant but is not the midpoint 0.

8. An instrument reports $\text{Arg}(z-2i)=0$. [6 marks]

(a) Describe the locus with all endpoint restrictions. [3 marks]

$z-2i$ must be positive real. It cannot be zero, so $z=2i$ is excluded.

Answer: The horizontal ray $y=2$, $x>0$.

(b) Find its intersection with $|z|=\sqrt{13}$. [3 marks]

With $y=2$, $x^2+4=13$ gives $x=\pm 3$. Only $x=3$ is on the required ray.

Answer: $z=3+2i$.

9. A real polynomial P has the non-real root $a+bi$, where a and b are real and $b\neq 0$. [6 marks]

(a) Explain why the conjugate is a root. [3 marks]

For real coefficients, $P(\overline{z})=\overline{P(z)}$. Conjugating $P(a+bi)=0$ gives the result.

Answer: $P(a-bi)=0$.

(b) Find the real quadratic factor contributed by the pair. [3 marks]

Multiply $(z-a-bi)(z-a+bi)=(z-a)^2+b^2$.

Answer: $z^2-2az+a^2+b^2$.

10. A student claims that every real cubic has exactly one real root and two non-real roots. [5 marks]

(a) Give a real cubic that disproves the claim. [2 marks]

It has three distinct real roots $-1, 0, 1$.

Answer: $P(z)=z(z-1)(z+1)$ is one example.

(b) State and justify the correct weaker conclusion for every real cubic. [3 marks]

Non-real roots occur in conjugate pairs, so all three roots cannot be non-real. This does not rule out three real roots.

Answer: Every real cubic has at least one real root, counting multiplicity.