Complex Numbers - Paper 5

10 questions | 56 marks | 70 minutes

Applications and geometry: transformations, distance constraints, phase relations and coefficient models.

Equivalent correct methods earn the corresponding marks.

1. A planar motion uses $T(z) = (2 + i)z$. [5 marks]

(a) Find $T(1 - i)$ in Cartesian form. [2 marks]

$(2 + i)(1 - i) = 2 - 2i + i - i^2 = 3 - i$.

Answer: $3 - i$.

(b) State the scale factor and give the rotation angle exactly using an inverse tangent. [3 marks]

The multiplier is in quadrant I, with modulus $\sqrt{4 + 1}$ and tangent 1/2.

Answer: Scale factor $\sqrt{5}$; anticlockwise angle $\tan^{-1}(1/2)$.

2. A complex signal filter acts by $w = z/(1 + i)$. The input is $z = 6i$. [5 marks]

(a) Find the output w. [2 marks]

$6i/(1 + i) = 6i(1 - i)/2 = 3 + 3i$.

Answer: $3 + 3i$.

(b) Compare output amplitude and phase with input amplitude and phase. [3 marks]

The filter multiplier is $(1/\sqrt{2})\text{cis}(-\pi/4)$. The input modulus 6 becomes $3\sqrt{2}$.

Answer: The amplitude is multiplied by $1/\sqrt{2}$ and the phase decreases by $\pi/4$.

3. A point z is moved by $w = iz + 2 - i$. [6 marks]

(a) Find the image of $z = 1 + 2i$. [2 marks]

$i(1 + 2i) + 2 - i = i - 2 + 2 - i = 0$.

Answer: $w = 0$.

(b) Find the fixed point, where w=z. [4 marks]

$(1 - i)z = 2 - i$, so $z = (2 - i)(1 + i)/2 = (3 + i)/2$.

Answer: $z = \frac{3}{2} + \frac{1}{2}i$.

4. A circular tracking zone has equation $|z - (2 + i)| = \sqrt{5}$. [6 marks]

(a) Find every intersection with the real axis. [3 marks]

Set $y = 0$ in $(x - 2)^2 + (y - 1)^2 = 5$; then $(x - 2)^2 = 4$.

Answer: $z = 0$ and $z = 4$.

(b) Find the greatest distance from the origin of any point in the closed zone $|z - (2 + i)| \le \sqrt{5}$. [3 marks]

The centre is $\sqrt{5}$ from the origin and the radius is $\sqrt{5}$. The farthest boundary point lies on the ray through the centre, at twice its position.

Answer: $2\sqrt{5}$.

5. Two beacons represent $a = -1 + i$ and $b = 3 + i$. A receiver is equally distant from both. [6 marks]

(a) Find the receiver's possible locus. [3 marks]

The beacon segment is horizontal with midpoint $1 + i$, so its perpendicular bisector is $x = 1$. Equivalently, square the equal distances.

Answer: $\operatorname{Re} z = 1$.

(b) Find its possible positions if it is also 2 units from i. [3 marks]

The circle $|z - i| = 2$ gives $1 + (y - 1)^2 = 4$, so $y = 1 \pm \sqrt{3}$. Add these imaginary coordinates to real part 1.

Answer: $z = 1 + (1 + \sqrt{3})i$ and $z = 1 + (1 - \sqrt{3})i$.

6. Consider $\operatorname{Arg}(z - (1 + i)) = -\pi/4$. [5 marks]

(a) Give a parametric description using one positive real parameter t. [3 marks]

The required direction is $1 - i$; start at $1 + i$ and add $t(1 - i)$. The endpoint is excluded.

Answer: $z = 1 + t + (1 - t)i$, $t > 0$.

(b) Find its intersection with the real axis. [2 marks]

The imaginary coordinate $1 - t$ is zero when $t = 1$, which is positive. Then the real coordinate is 2.

Answer: $z = 2$.

7. The map $w = (1 + i)z$ acts on the circle $|z - 2| = 1$. [5 marks]

(a) State the centre and radius of the image circle. [3 marks]

The original centre maps to $(1 + i)2$. Every displacement from the centre is multiplied in modulus by $|1 + i| = \sqrt{2}$.

Answer: Centre $2 + 2i$; radius $\sqrt{2}$.

(b) Write a modulus equation for the image circle. [2 marks]

Use the image centre and radius, or multiply $|z - 2| = 1$ by $|1 + i|$.

Answer: $|w - (2 + 2i)| = \sqrt{2}$.

8. A repeated steering instruction multiplies a position by $q = \text{cis}(2\pi/9)$. [6 marks]

(a) Find the least positive number of instructions returning every position to its original value. [3 marks]

Require $q^n = 1$, so $2n\pi/9$ must be a multiple of 2π. The least positive n is 9.

Answer: 9.

(b) The instruction instead uses $2q$. Can a nonzero point ever return exactly after a positive number of repetitions? [3 marks]

After n repetitions the modulus is multiplied by $2^n > 1$, so a nonzero point cannot recover its original modulus.

Answer: No.

9. A real monic polynomial has degree 4 and roots $2 + i$ and $2 + 3i$. [6 marks]

(a) Give its factorisation into real quadratics. [3 marks]

Include the conjugates $2 - i$ and $2 - 3i$, then form the two pair factors.

Answer: $(z^2 - 4z + 5)(z^2 - 4z + 13)$.

(b) Find its constant term and coefficient of z^3. [3 marks]

The constant is 5×13. The cubic coefficient is the sum of the quadratic linear coefficients, $-4 - 4$.

Answer: Constant term 65; coefficient of z^3 is -8.

10. An algorithm proposes two solutions of $z^2 = 5 + 12i$: $3 + 2i$ and $3 - 2i$. [6 marks]

(a) Identify the invalid solution and replace it. [3 marks]

$(3 + 2i)^2 = 5 + 12i$, but $(3 - 2i)^2 = 5 - 12i$. The second square root is the negative of the first.

Answer: $3 - 2i$ is invalid; replace it by $-3 - 2i$.

(b) Explain why conjugating one root did not give the other root. [3 marks]

The real-coefficient conjugate-root theorem does not apply; a nonzero square has roots that are negatives of each other.

Answer: The equation is $z^2 - (5 + 12i) = 0$, whose constant coefficient is not real.