Complex Numbers - Paper 6

10 questions | 68 marks | 75 minutes

A challenging synthesis paper with exact roots, geometric restrictions, proof and polynomial reconstruction.

Equivalent correct methods earn the corresponding marks.

1. Let $z = -1 + \sqrt{3}i$. [6 marks]

(a) Find its principal-argument polar form and the principal argument of its inverse. [3 marks]

The point is in quadrant II; taking a reciprocal negates the argument.

Answer: $z = 2\text{cis}(2\pi/3)$; $\text{Arg}(1/z) = -2\pi/3$.

(b) Calculate z^5 exactly. [3 marks]

De Moivre gives $32\text{cis}(10\pi/3) = 32\text{cis}(-2\pi/3)$.

Answer: $-16 - 16\sqrt{3}i$.

2. Two nonzero complex numbers satisfy $|u| = 3$, $|v| = 2$, $\text{Arg}\, u = 5\pi/6$, and $\text{Arg}\, v = -2\pi/3$. [6 marks]

(a) Find the principal-argument polar form of u/v. [3 marks]

Subtract angles: $5\pi/6 + 2\pi/3 = 3\pi/2$, which reduces to $-\pi/2$.

Answer: $\frac{3}{2}\text{cis}(-\pi/2)$.

(b) Find $|u - v|$ exactly. [3 marks]

The angle difference has cosine 0. Thus $|u - v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos(3\pi/2) = 9 + 4 = 13$.

Answer: $\sqrt{13}$.

3. Let $z = x + iy$. Consider $|z - 1| = 2|z + 1|$. [6 marks]

(a) Find its Cartesian equation in completed-square form. [4 marks]

Square: $(x-1)^2 + y^2 = 4[(x+1)^2 + y^2]$. Rearrange to $3x^2 + 10x + 3y^2 + 3 = 0$, then complete the square.

Answer: $(x + 5/3)^2 + y^2 = 16/9$.

(b) Give the centre and radius and explain why the locus is not a line. [2 marks]

The unequal distance ratio leaves quadratic terms after expansion; it is a circle rather than an equal-distance bisector.

Answer: Centre $(-5/3, 0)$; radius $4/3$.

4. Solve $(z-1)^3 = 8i$. [6 marks]

(a) List all solutions in exact Cartesian form. [4 marks]

Let $w = z - 1$. The roots of $w^3 = 8i$ have modulus 2 and arguments $\pi/6, 5\pi/6, 3\pi/2$. Convert and add 1.

Answer: $z = 1 + \sqrt{3} + i,\ 1 - \sqrt{3} + i,\ 1 - 2i$.

(b) State the centre of the triangle formed by the roots and justify it. [2 marks]

The w-roots form a regular triangle centred at zero, and the map $z = w + 1$ translates every point by 1.

Answer: The centre is $1 + 0i$.

5. Let $\zeta = \text{cis}(2\pi/5)$. [8 marks]

(a) Show that $\zeta + \zeta^{-1} = 2\cos(2\pi/5)$ and is real. [2 marks]

The two terms are conjugates on the unit circle, so their imaginary parts cancel.

Answer: $\zeta + \zeta^{-1} = 2\cos(2\pi/5)$.

(b) Put $t = \zeta + \zeta^{-1}$. Prove $t^2 + t - 1 = 0$. [4 marks]

Divide $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$ by ζ^2: $(\zeta^2 + \zeta^{-2}) + (\zeta + \zeta^{-1}) + 1 = 0$. Since $\zeta^2 + \zeta^{-2} = t^2 - 2$, obtain the quadratic.

Answer: $t^2 + t - 1 = 0$.

(c) Hence find $\cos(2\pi/5)$ exactly. [2 marks]

$t = (-1 \pm \sqrt{5})/2$; $\cos(2\pi/5) > 0$, so take the positive t and divide by 2.

Answer: $\frac{\sqrt{5}-1}{4}$.

6. The locus $|z| = 1$ is restricted by $\text{Arg}\left(\frac{z-1}{z+1}\right) = \pi/2$. [7 marks]

(a) For $z = x + iy$ on the circle with $z \neq -1$, show that $\frac{z-1}{z+1} = \frac{2iy}{(x+1)^2 + y^2}$. [4 marks]

Multiply by $\overline{z+1} = x + 1 - iy$. The numerator is $x^2 + y^2 - 1 + 2iy = 2iy$ on the unit circle.

Answer: The quotient has the stated pure-imaginary form.

(b) Describe exactly which part of the unit circle satisfies the argument condition. [3 marks]

The denominator above is positive where defined, so the quotient has argument $\pi/2$ exactly when $y > 0$. At $z = -1$ it is undefined; at $z = 1$ it is zero with undefined argument.

Answer: The open upper semicircle, $|z| = 1$ with $y > 0$; both real endpoints are excluded.

7. Consider $P(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$. [6 marks]

(a) Show that $1 + 2i$ is a root by identifying a real quadratic factor, and find the quotient. [3 marks]

The pair $1 \pm 2i$ yields $z^2 - 2z + 5$; multiplication by $z^2 + 1$ gives the stated P.

Answer: $P(z) = (z^2 - 2z + 5)(z^2 + 1)$.

(b) Factorise P completely over the complex numbers. [3 marks]

The quotient has roots $\pm i$ and the first factor has roots $1 \pm 2i$.

Answer: $(z - (1 + 2i))(z - (1 - 2i))(z - i)(z + i)$.

8. A real monic cubic has roots $a + bi$, $a - bi$ and a real number r, with $b \neq 0$. Its coefficient of z^2 is 0 and its constant term is -20. Also $a = 1$. [8 marks]

(a) Find r and explain whether a real value of b can satisfy all conditions. [4 marks]

The root sum is $2a + r = 0$, so $r = -2$. The constant is $-r(a^2 + b^2) = 2(1 + b^2) > 0$, which cannot be -20.

Answer: $r = -2$; no real b can satisfy the constant-term condition.

(b) If the constant term were +20 instead, find b and the cubic. [4 marks]

$2(1 + b^2) = 20$ gives $b^2 = 9$. Multiply $(z + 2)(z^2 - 2z + 10)$ to obtain $z^3 + 6z + 20$.

Answer: $b = \pm 3$; $P(z) = z^3 + 6z + 20$.

9. A map is $T(z) = az + b$. It sends 0 to $1 - i$ and 1 to $3 + i$. [8 marks]

(a) Find a and b, and describe the scale and turn in its multiplication step. [4 marks]

$T(0) = b$. Then $a + b = 3 + i$, giving $a = 2 + 2i$. Express a in polar form.

Answer: $b = 1 - i$, $a = 2 + 2i$; scale $2\sqrt{2}$, anticlockwise rotation $\pi/4$.

(b) Find the unique fixed point of T. [4 marks]

From $z = (2 + 2i)z + 1 - i$, obtain $(-1 - 2i)z = 1 - i$. Divide with the conjugate: $z = (1 - i)(-1 + 2i)/5 = (1 + 3i)/5$.

Answer: $z = \frac{1+3i}{5}$.

10. Let n be an integer greater than 1 and $\omega = \text{cis}(2\pi/n)$. [7 marks]

(a) Prove that the product of all n roots of unity is $(-1)^{n-1}$. [4 marks]

The product is $\omega^{0+1+\cdots+(n-1)} = \text{cis}(\pi(n-1)) = (-1)^{n-1}$.

Answer: The product is $(-1)^{n-1}$.

(b) Explain why conjugate pairing gives the same result, distinguishing even and odd n. [3 marks]

All roots have modulus 1, so $z\bar{z} = 1$. The remaining real factors determine the sign.

Answer: Each non-real conjugate pair has product 1. For odd n the only real root is 1, so the product is 1; for even n the real roots are 1 and -1, so the product is -1.