Complex Numbers - Paper 6

Name: _____ Date: _____

10 questions | 68 marks | 75 minutes

A challenging synthesis paper with exact roots, geometric restrictions, proof and polynomial reconstruction.

A scientific calculator may be used.

1. Let $z = -1 + \sqrt{3}i$. [6 marks]

(a) Find its principal-argument polar form and the principal argument of its inverse. [3 marks]

(b) Calculate z^5 exactly. [3 marks]

2. Two nonzero complex numbers satisfy $|u| = 3$, $|v| = 2$, $\text{Arg}\, u = 5\pi/6$, and $\text{Arg}\, v = -2\pi/3$. [6 marks]

(a) Find the principal-argument polar form of u/v. [3 marks]

(b) Find $|u - v|$ exactly. [3 marks]

3. Let $z = x + iy$. Consider $|z - 1| = 2|z + 1|$. [6 marks]

(a) Find its Cartesian equation in completed-square form. [4 marks]

(b) Give the centre and radius and explain why the locus is not a line. [2 marks]

4. Solve $(z-1)^3 = 8i$. [6 marks]

(a) List all solutions in exact Cartesian form. [4 marks]

(b) State the centre of the triangle formed by the roots and justify it. [2 marks]

5. Let $\zeta = \text{cis}(2\pi/5)$. [8 marks]

(a) Show that $\zeta + \zeta^{-1} = 2\cos(2\pi/5)$ and is real. [2 marks]

(b) Put $t = \zeta + \zeta^{-1}$. Prove $t^2 + t - 1 = 0$. [4 marks]

(c) Hence find $\cos(2\pi/5)$ exactly. [2 marks]

6. The locus $|z| = 1$ is restricted by $\text{Arg}\left(\frac{z-1}{z+1}\right) = \pi/2$. [7 marks]

(a) For $z = x + iy$ on the circle with $z \neq -1$, show that $\frac{z-1}{z+1} = \frac{2iy}{(x+1)^2 + y^2}$. [4 marks]

(b) Describe exactly which part of the unit circle satisfies the argument condition. [3 marks]

7. Consider $P(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$. [6 marks]

(a) Show that $1 + 2i$ is a root by identifying a real quadratic factor, and find the quotient. [3 marks]

(b) Factorise P completely over the complex numbers. [3 marks]

8. A real monic cubic has roots $a + bi$, $a - bi$ and a real number r, with $b \neq 0$. Its coefficient of z^2 is 0 and its constant term is -20. Also $a = 1$. [8 marks]

(a) Find r and explain whether a real value of b can satisfy all conditions. [4 marks]

(b) If the constant term were +20 instead, find b and the cubic. [4 marks]

9. A map is $T(z) = az + b$. It sends 0 to $1 - i$ and 1 to $3 + i$. [8 marks]

(a) Find a and b, and describe the scale and turn in its multiplication step. [4 marks]

(b) Find the unique fixed point of T. [4 marks]

10. Let n be an integer greater than 1 and $\omega = \text{cis}(2\pi/n)$. [7 marks]

(a) Prove that the product of all n roots of unity is $(-1)^{n-1}$. [4 marks]

(b) Explain why conjugate pairing gives the same result, distinguishing even and odd n. [3 marks]