Functions and graph sketching - Practice test 1

10 questions | 46 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = \sqrt{x+2}$ and $g(x) = 3x - 1$. [4 marks]

(a) Find $(f \circ g)(x)$ and its domain. [2 marks]

The inner line is defined for all real inputs. The outer root requires $3x - 1 + 2 \geq 0$.

Answer: $\sqrt{3x+1}$, $x \geq -\frac{1}{3}$.

(b) Find $(g \circ f)(x)$ and its domain. Are the two functions equal? [2 marks]

The root now acts first and requires $x \geq -2$. The rules and domains differ.

Answer: $3\sqrt{x+2} - 1$, $x \geq -2$; they are not equal.

2. Consider $f(x) = x^2 - 4x + 1$ on the real numbers. [4 marks]

(a) Use two distinct inputs to show that f is not one-to-one. [2 marks]

The inputs 1 and 3 are both allowed and unequal.

Answer: For example, $f(1) = f(3) = -2$.

(b) Give the largest interval starting at the vertex on which f is increasing, and state its range. [2 marks]

Complete the square: $f(x) = (x-2)^2 - 3$. The right branch increases from its minimum.

Answer: Domain $[2, \infty)$; range $[-3, \infty)$.

3. Let $f(x) = 5 - 2x$ on the real numbers. [4 marks]

(a) Find f^{-1} and verify one of the inverse compositions. [2 marks]

From $y = 5 - 2x$, solve $x = (5 - y)/2$. Substitute: $5 - 2((5 - x)/2) = x$.

Answer: $f^{-1}(x) = (5 - x)/2$; $f(f^{-1}(x)) = x$.

(b) Find the point where the graph of f meets the graph of its inverse. [2 marks]

Solve the two line equations together: $5 - 2x = (5 - x)/2$ gives $x = 5/3$, and substitution gives $y = 5/3$.

Answer: $(\frac{5}{3}, \frac{5}{3})$.

4. Let $h(x) = |2x + 2| - 4$. [5 marks]

(a) Write h piecewise and find its vertex. [2 marks]

The expression inside the absolute value changes sign at $x = -1$.

Answer: $h(x) = -2x - 6$ for $x < -1$, and $h(x) = 2x - 2$ for $x \geq -1$; vertex $(-1, -4)$.

(b) Find the intercepts and solve $h(x) < 0$. [3 marks]

$|2x + 2| = 4$ gives $x = -3, 1$; substitute zero for the vertical intercept. The graph is below zero between its roots.

Answer: Intercepts $(-3, 0), (1, 0), (0, -2)$; $-3 < x < 1$.

5. Let $r(x) = 1/(x-2)^2$. [5 marks]

(a) State its domain, asymptotes and range. [3 marks]

The squared denominator is positive away from 2, can be arbitrarily small, and grows without bound.

Answer: Domain $x \neq 2$; asymptotes $x = 2$, $y = 0$; range $(0, \infty)$.

(b) State its symmetry and solve $r(x) = 1$. [2 marks]

Inputs equally far from 2 have equal squares. Solve $(x-2)^2 = 1$.

Answer: Symmetric about $x = 2$; $x = 1$ or $x = 3$.

6. Consider $q(x) = (x^2 - 9)/(x-3)$. [4 marks]

(a) Simplify the rule while retaining the original domain. [2 marks]

Factor the numerator as $(x-3)(x+3)$; cancellation is valid only when $x - 3 \neq 0$.

Answer: $q(x) = x + 3$ for $x \neq 3$.

(b) Give the hole and both intercepts. [2 marks]

The missing input is 3, whose line value would be 6. The root -3 and input 0 are both permitted.

Answer: Hole $(3, 6)$; intercepts $(-3, 0)$ and $(0, 3)$.

7. For $f(x) = x + 4/x$, with $x \neq 0$: [6 marks]

(a) Find and classify the stationary points. [3 marks]

$f'(x) = 1 - 4/x^2$. The derivative is positive for $|x| > 2$ and negative for $0 < |x| < 2$.

Answer: Local maximum $(-2, -4)$; local minimum $(2, 4)$.

(b) Give the asymptotes and justify the complete range. [3 marks]

For $x > 0$, $x + 4/x \geq 4$, and the positive branch attains every value from 4 upward. Odd symmetry gives the negative branch.

Answer: Asymptotes $x = 0$, $y = x$; range $(-\infty, -4] \cup [4, \infty)$.

8. Let $u(x) = 1/(x^2 + 4)$. [5 marks]

(a) Use differentiation and symmetry to locate its maximum. [2 marks]

$u'(x) = -2x/(x^2 + 4)^2$, positive for $x < 0$ and negative for $x > 0$.

Answer: Global maximum $(0, \frac{1}{4})$; the graph is even.

(b) Describe a complete sketch, including intercepts and end behaviour. [3 marks]

The denominator is never zero, so there is no vertical asymptote. Label the maximum and the horizontal asymptote.

Answer: No horizontal-axis intercept; vertical intercept $\frac{1}{4}$; all real inputs; positive branches approach $y = 0$ at both ends.

9. A sensor uses output $f(t) = \sqrt{t - 2}$ for time $t \geq 2$. A clock display reports $t = 3s + 2$. [5 marks]

(a) Express the output in s and give its domain. [3 marks]

Require $3s + 2 \geq 2$.

Answer: $\sqrt{3s}$, $s \geq 0$.

(b) Find the displayed s when output is 6. [2 marks]

$\sqrt{3s} = 6$ gives 3s=36.

Answer: s=12.

10. A student states that $(x^2 - 1)/(x - 1)$ and x+1 are the same function on all real inputs. [4 marks]

(a) Identify the false part of the claim. [2 marks]

The original denominator is zero there.

Answer: The quotient excludes x=1.

(b) Describe the graphical difference. [2 marks]

The line includes this point.

Answer: The quotient has a hole at (1,2).