Functions and graph sketching - Practice test 2

10 questions | 48 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = 1/(x-1)$ and $g(x) = \sqrt{x}$. [4 marks]

(a) Find $(f \circ g)(x)$ and its domain. [2 marks]

The root requires $x \geq 0$; its value must also differ from 1.

Answer: $1/(\sqrt{x}-1)$, $x \geq 0$ and $x \neq 1$.

(b) Find $(g \circ f)(x)$ and its domain. [2 marks]

The reciprocal is defined when $x \neq 1$. Its value is nonnegative only when $x > 1$; it cannot equal zero.

Answer: $\sqrt{1/(x-1)}$, $x > 1$.

2. Consider $f(x) = (x-1)^2$ on the restricted domain $[-1, 1]$. [5 marks]

(a) Prove that this restriction is one-to-one. [2 marks]

For inputs at most 1, increasing the input decreases its nonnegative distance from 1, and hence decreases its square. Equivalently, $f'(x) = 2(x-1) < 0$ in the interval interior.

Answer: It is strictly decreasing on $[-1, 1]$.

(b) Find the inverse, including its domain and range. [3 marks]

The original input satisfies $x - 1 \leq 0$, selecting the negative root. The original endpoint values are 4 and 0.

Answer: $f^{-1}(x) = 1 - \sqrt{x}$; domain $[0, 4]$, range $[-1, 1]$.

3. Let $f(x) = (2x + 3)/(x - 1)$, $x \neq 1$. [4 marks]

(a) Find its inverse and excluded inverse input. [2 marks]

$y(x - 1) = 2x + 3$ gives $x(y - 2) = y + 3$.

Answer: $f^{-1}(x) = (x + 3)/(x - 2)$, $x \neq 2$.

(b) Find $f^{-1}(-3)$ and verify the result in the original rule. [2 marks]

Substitution gives $(-3 + 3)/(-3 - 2) = 0$, then $3/(-1) = -3$.

Answer: $f^{-1}(-3) = 0$ and $f(0) = -3$.

4. Let $v(x) = |x^2 - 9|$. [5 marks]

(a) Find the zeros, local extrema and range. [3 marks]

Between -3 and 3 the graph is $9 - x^2$; outside, it is $x^2 - 9$.

Answer: Zeros ± 3; corner minima $(\pm 3, 0)$; local maximum $(0, 9)$; range $[0, \infty)$.

(b) Find all intersections with $y = 5$. [2 marks]

$x^2 - 9 = 5$ gives $x^2 = 14$; $x^2 - 9 = -5$ gives $x^2 = 4$.

Answer: $x = \pm 2, \ \pm\sqrt{14}$.

5. Let $r(x) = -3/(x + 2) + 1$. [5 marks]

(a) Find its asymptotes, centre of symmetry and intercepts. [3 marks]

Set the rule equal to zero for the root; substitute $x = 0$ for the vertical intercept.

Answer: Asymptotes $x = -2$, $y = 1$; centre $(-2, 1)$; intercepts $(1, 0)$ and $(0, -\frac{1}{2})$.

(b) Determine where the graph is increasing and explain its range. [2 marks]

$r' = 3/(x + 2)^2 > 0$ wherever defined. The nonzero reciprocal term can have every nonzero real value.

Answer: Increasing on each of $(-\infty, -2)$ and $(-2, \infty)$; range $\mathbb{R} \setminus \{1\}$.

6. Let $f(x) = (x+2)/(x^2+x-2)$. [5 marks]

(a) Factor, simplify and locate every discontinuity. [3 marks]

The denominator is $(x+2)(x-1)$. Keep both original exclusions.

Answer: $f(x) = 1/(x-1)$ for $x \neq -2, 1$; hole $(-2, -\frac{1}{3})$ and vertical asymptote $x = 1$.

(b) Find the range, taking account of the hole. [2 marks]

The reciprocal rule never equals zero. It attains $-\frac{1}{3}$ only at the excluded input -2.

Answer: $\mathbb{R} \setminus \{0, -\frac{1}{3}\}$.

7. Let $f(x) = x^2/(x+2)$. [5 marks]

(a) Use division to find its asymptotes. [2 marks]

Divide the numerator by $x+2$; the remainder is 4.

Answer: $f(x) = x - 2 + 4/(x+2)$; asymptotes $x = -2$, $y = x - 2$.

(b) Find and classify its stationary points. [3 marks]

$f' = x(x+4)/(x+2)^2$. Its sign is positive before -4, negative between -4 and 0 apart from the pole, and positive after 0.

Answer: Local maximum $(-4, -8)$; local minimum $(0, 0)$.

8. Let $f(x) = (x^2 + 1)/(x^2 + 4)$. [5 marks]

(a) Find its symmetry, stationary point and range. [3 marks]

$f = 1 - 3/(x^2 + 4)$ and $f' = 6x/(x^2 + 4)^2$. The minimum occurs at zero and 1 is approached from below.

Answer: Even; global minimum $(0, \frac{1}{4})$; range $[\frac{1}{4}, 1)$.

(b) Give all intercepts and asymptotes needed for a sketch. [2 marks]

Neither numerator nor denominator has a real zero.

Answer: Vertical intercept $\frac{1}{4}$; no horizontal-axis intercept; horizontal asymptote $y = 1$; no vertical asymptote.

9. A machine is calibrated by $y = (x - 4)^2 + 1$ on $x \geq 4$. [5 marks]

(a) Find the inverse calibration and its domain. [3 marks]

Choose the nonnegative square-root branch.

Answer: $x = 4 + \sqrt{y - 1}$, $y \geq 1$.

(b) Find the input producing output 10. [2 marks]

$4 + \sqrt{9} = 7$.

Answer: x=7.

10. A sketch of $1/(x^2 + 1)$ shows two vertical asymptotes. [5 marks]

(a) Explain why that cannot be correct. [2 marks]

x squared plus one is always at least one.

Answer: The real denominator never vanishes.

(b) State its actual domain, range and end behaviour. [3 marks]

The maximum occurs at zero.

Answer: Domain real; range $(0, 1]$; approaches zero at both ends.