Functions and graph sketching - Practice test 3

10 questions | 49 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = \ln x$ and $g(x) = x^2 - 4$. [4 marks]

(a) Find the rule and domain of $f \circ g$. [2 marks]

The logarithm argument must be strictly positive, so $x^2 > 4$.

Answer: $\ln(x^2 - 4)$ for $x < -2$ or $x > 2$.

(b) Find the rule and domain of $g \circ f$ and compare its symmetry with the first composition. [2 marks]

The inner logarithm excludes all nonpositive inputs. Replacing x with $-x$ is therefore not permitted throughout this domain.

Answer: $(\ln x)^2 - 4$, $x > 0$. The first composition is even; this second one is not even on its domain.

2. Let $f(x) = x^3 - 3x$. [5 marks]

(a) Find its intervals of increase and decrease. [3 marks]

$f' = 3(x^2 - 1)$ is positive outside $[-1, 1]$ and negative inside.

Answer: Increasing on $(-\infty, -1]$ and $[1, \infty)$; decreasing on $[-1, 1]$.

(b) How many solutions does $f(x) = 1$ have? Justify without finding the roots. [2 marks]

The local maximum is 2 and the local minimum is -2. The line $y = 1$ crosses each of the three monotonic branches.

Answer: Three distinct real solutions.

3. Let $f(x) = 2e^{x-1} + 3$. [5 marks]

(a) Find its inverse, domain and range. [3 marks]

Solve $y - 3 = 2e^{x-1}$ for the original input; the logarithm argument must be positive.

Answer: $f^{-1}(x) = 1 + \ln((x-3)/2)$; domain $(3, \infty)$ and range $\mathbb{R}$.

(b) Find the exact input for output 7. [2 marks]

$2e^{x-1} + 3 = 7$ gives $e^{x-1} = 2$.

Answer: $x = 1 + \ln 2$.

4. Consider $y = |x - 1| + |x + 1|$. [5 marks]

(a) Write a piecewise rule on the three intervals separated by -1 and 1. [3 marks]

Choose each absolute-value sign according to its inside expression.

Answer: $y = -2x$ for $x < -1$; $y = 2$ for $-1 \le x \le 1$; $y = 2x$ for $x > 1$.

(b) Find the range and all solutions of $y = 2$ and $y = 4$. [2 marks]

The middle segment is horizontal. Use the two outer linear branches for output 4.

Answer: Range $[2, \infty)$; $y = 2$ for $[-1, 1]$; $y = 4$ at $x = \pm 2$.

5. Let $r(x) = 1/(x^2 - 4)$. [5 marks]

(a) Find the asymptotes and classify its stationary point. [3 marks]

$r' = -2x/(x^2 - 4)^2$, changing from positive to negative at 0.

Answer: Asymptotes $x = \pm 2$, $y = 0$; local maximum $(0, -\frac{1}{4})$.

(b) Find all points fixed by replacing the ordinate $r(x)$ with its reciprocal. [2 marks]

Fixed nonzero ordinates are ± 1. Solve $x^2 - 4 = 1$ or $x^2 - 4 = -1$.

Answer: $(\pm\sqrt{5}, 1)$ and $(\pm\sqrt{3}, -1)$.

6. Let $f(x) = (x^2 - 1)/(x^2 + x)$. [5 marks]

(a) Simplify and locate the hole and pole. [3 marks]

Factor numerator and denominator, cancel $x + 1$, and retain both exclusions.

Answer: $f = (x - 1)/x$ for $x \neq -1, 0$; hole $(-1, 2)$; pole at $x = 0$.

(b) Find both intercepts and the full range. [2 marks]

The rule $1 - 1/x$ never equals 1. It equals 2 only at the hole input -1.

Answer: Horizontal intercept $(1, 0)$; no vertical intercept; range $\mathbb{R} \setminus \{1, 2\}$.

7. For $f(x) = x/(x^2 + 4)$: [5 marks]

(a) Find and classify all stationary points. [3 marks]

$f' = (4 - x^2)/(x^2 + 4)^2$. Use its signs outside and inside $[-2, 2]$.

Answer: Local minimum $(-2, -\frac{1}{4})$ and local maximum $(2, \frac{1}{4})$.

(b) Find every solution of $f(x) = \frac{1}{5}$. [2 marks]

$5x = x^2 + 4$ gives $(x - 1)(x - 4) = 0$; neither input is excluded.

Answer: $x = 1$ or $x = 4$.

8. Let $g(x) = x^3/(x^2 + 1)$. [5 marks]

(a) Find its asymptote and classify the stationary point at zero. [3 marks]

$g = x - x/(x^2 + 1)$ and $g' = x^2(x^2 + 3)/(x^2 + 1)^2$. The derivative is positive on both sides of zero; locally the leading behaviour is x^3.

Answer: Oblique asymptote $y = x$; stationary inflection $(0, 0)$.

(b) Explain why the graph is one-to-one even though $g'(0) = 0$. [2 marks]

The derivative is positive except at the single input zero; no interval is flat and no output repeats.

Answer: It is strictly increasing on the whole real line.

9. A charge model is $C(x) = x + 4/x$ dollars, with batch size $x > 0$. [6 marks]

(a) Find the stationary point. [3 marks]

$C'(x) = 1 - 4/x^2 = 0$ gives x=2 in the domain.

Answer: (2,4).

(b) Show it is a minimum and interpret it. [3 marks]

Derivative is negative before 2 and positive after 2.

Answer: Minimum charge 4 dollars at batch size 2.

10. For $f(x) = x - 3$, a student says $f(|x|)$ is never negative. [4 marks]

(a) Give a counterexample. [2 marks]

Evaluate |0|-3.

Answer: x=0 gives -3.

(b) Explain the confusion. [2 marks]

Only the latter forces nonnegative output.

Answer: The student confused f(|x|) with |f(x)|.