Functions and graph sketching - Practice test 4

10 questions | 48 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = \sqrt{4 - x}$ and $g(x) = 1/x$. [5 marks]

(a) Find the domain of $f \circ g$. [3 marks]

Require $x \neq 0$ and $4 - 1/x \geq 0$. Every negative input works; for positive inputs multiply by x to obtain $x \geq 1/4$.

Answer: $(-\infty, 0) \cup [\frac{1}{4}, \infty)$.

(b) Find the domain of $g \circ f$. Why does an endpoint disappear? [2 marks]

The root accepts $x \leq 4$, but at 4 it outputs zero, which the reciprocal cannot accept.

Answer: $x < 4$.

2. Consider the family $f_a(x) = ax^2 + 2x$, with real a. [5 marks]

(a) For which a is f_a one-to-one on the entire real line? [2 marks]

When $a = 0$, the function is the line $2x$. If $a \neq 0$, its quadratic graph is symmetric about $x = -1/a$ and fails the horizontal-line test.

Answer: Only $a = 0$.

(b) For $a = 1$, restrict to the increasing branch and give its inverse. [3 marks]

$x^2 + 2x = (x + 1)^2 - 1$; choose the nonnegative square-root branch.

Answer: Domain $[-1, \infty)$; inverse $-1 + \sqrt{x + 1}$ with inverse domain $[-1, \infty)$.

3. Let $f(x) = x/(x-1)$ on $\mathbb{R} \setminus \{1\}$. [4 marks]

(a) Show that f is its own inverse. [2 marks]

$f(f(x)) = [x/(x-1)]/[x/(x-1) - 1] = x$. The inner value cannot equal 1.

Answer: $f(f(x)) = x$ for every $x \neq 1$.

(b) Find its real fixed points and explain their relation to $y = x$. [2 marks]

Solve $x/(x-1) = x$: $x(x-2) = 0$. Both are allowed, and each lies on the reflection line.

Answer: $(0, 0)$ and $(2, 2)$.

4. Let $f(x) = |x^2 - 2x|$. [5 marks]

(a) Give its piecewise rule and all local extrema. [3 marks]

The quadratic inside is nonpositive on $[0, 2]$ and nonnegative outside.

Answer: $x^2 - 2x$ for $x \leq 0$ or $x \geq 2$, and $2x - x^2$ for $0 \leq x \leq 2$; minima $(0, 0)$, $(2, 0)$ and local maximum $(1, 1)$.

(b) For which real c does $f(x) = c$ have four distinct solutions? [2 marks]

Each positive level below the middle maximum meets the two inner arcs and the two outer branches. At $c = 1$ the two inner intersections merge.

Answer: $0 < c < 1$.

5. Let $r(x) = 1/(x^2 + 2x + 2)$. [5 marks]

(a) Find its domain, symmetry, maximum and range. [3 marks]

Complete the square in the denominator: $(x+1)^2 + 1$.

Answer: All real inputs; symmetric about $x = -1$; maximum $(-1, 1)$; range $(0, 1]$.

(b) Find all intersections of the graph with $y = \frac{1}{2}$. [2 marks]

The denominator must be 2, so $(x+1)^2 = 1$.

Answer: $(-2, \frac{1}{2})$ and $(0, \frac{1}{2})$.

6. For $f(x) = (x^3 - x)/(x^2 - 1)$: [4 marks]

(a) Simplify, state the domain, and identify every hole. [2 marks]

Factor the numerator as $x(x-1)(x+1)$ and keep the original denominator exclusions.

Answer: $f(x) = x$ for $x \neq \pm 1$; holes $(-1, -1)$ and $(1, 1)$.

(b) Find the range and decide whether the function is one-to-one. [2 marks]

On the allowed domain the output equals the input, so neither omitted output can be obtained by another input.

Answer: Range $\mathbb{R} \setminus \{-1, 1\}$; it is one-to-one.

7. Let $f(x) = (x^2 - 1)/(x^2 + 1)$. [5 marks]

(a) Find its intercepts, stationary point and range. [3 marks]

$f = 1 - 2/(x^2 + 1)$ and $f' = 4x/(x^2 + 1)^2$.

Answer: Intercepts $(\pm 1, 0), (0, -1)$; global minimum $(0, -1)$; range $[-1, 1)$.

(b) How many intersections with a horizontal line $y = c$ occur for $-1 < c < 1$? Justify. [2 marks]

The graph is even and strictly decreases to its minimum then strictly increases; each branch approaches 1.

Answer: Two, one on each side of zero.

8. A rational graph has rule $y = x + 4/(x - 1)$. [6 marks]

(a) Find its asymptotes and stationary points. [3 marks]

$y' = 1 - 4/(x-1)^2 = 0$ gives $x = -1, 3$. Its signs classify the points.

Answer: Asymptotes $x = 1$, $y = x$; local maximum $(-1, -3)$ and local minimum $(3, 5)$.

(b) Justify its range using a quadratic discriminant. [3 marks]

$y(x-1) = x(x-1) + 4$ gives $x^2 - (y+1)x + y + 4 = 0$. Its discriminant is $(y-5)(y+3)$, which must be nonnegative.

Answer: $(-\infty, -3] \cup [5, \infty)$.

9. The height model $h(x) = |x - 4| + 2$ is used on $0 \le x \le 8$. [5 marks]

(a) Find the minimum and its location. [2 marks]

The absolute-value part is smallest at zero.

Answer: Minimum 2 at x=4.

(b) For which inputs is height 5? [3 marks]

Solve $|x - 4| = 3$ and check both inputs lie in the domain.

Answer: x=1 or x=7.

10. A graph of $x/(x^2 + 1)$ is said never to cross y=0 because y=0 is its asymptote. [4 marks]

(a) Refute the claim. [2 marks]

The denominator at zero is 1, so this input is valid.

Answer: f(0)=0.

(b) State what a horizontal asymptote actually describes. [2 marks]

It does not forbid intersections at finite inputs.

Answer: The limit at large positive or negative input magnitude.