# Functions and graph sketching - Practice test 5

10 questions | 51 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

**1.** Let $f(x) = \ln x$ and $g(x) = 1 - x^2$. [5 marks]

**(a)** Find the domains of both compositions. [2 marks]

The first logarithm needs $1 - x^2 > 0$; in the second composition the inner logarithm alone restricts the input.

**Answer:** $f \circ g$: $-1 < x < 1$; $g \circ f$: $x > 0$.

**(b)** Find and compare their ranges. [3 marks]

On $(-1, 1)$, $1 - x^2$ ranges over $(0, 1]$. Meanwhile $\ln x$ ranges over all real numbers, so $1 - (\ln x)^2 \leq 1$.

**Answer:** $f \circ g$ has range $(-\infty, 0]$; $g \circ f$ has range $(-\infty, 1]$.

**2.** Let $f(x) = x^3 - 3x$ restricted to $[-1, 1]$. [5 marks]

**(a)** Justify the existence of an inverse and state its domain and range, without finding an explicit rule. [3 marks]

$f' = 3x^2 - 3 < 0$ in the interior; $f(-1) = 2$ and $f(1) = -2$.

**Answer:** The function is strictly decreasing; inverse domain $[-2, 2]$, inverse range $[-1, 1]$.

**(b)** Find the inverse images of $-2, 0, 2$. [2 marks]

Use the values of the original function and the restricted domain.

**Answer:** $f^{-1}(-2) = 1$, $f^{-1}(0) = 0$, $f^{-1}(2) = -1$.

**3.** Let $f(x) = 4 - \sqrt{2x+1}$ with its largest real domain. [5 marks]

**(a)** Find the inverse and its domain and range. [3 marks]

The original root requires $x \geq -1/2$ and gives outputs at most 4. Solve $\sqrt{2x+1} = 4 - y$, keeping $y \leq 4$ before squaring.

**Answer:** $f^{-1}(x) = ((4-x)^2 - 1)/2$; domain $(-\infty, 4]$, range $[-\frac{1}{2}, \infty)$.

**(b)** Explain why the inverse formula must not be used at input 5. [2 marks]

Squaring the rearranged equation alone loses the condition $4 - y \geq 0$.

**Answer:** 5 lies outside the inverse domain because the original function never outputs 5.

**4.** Let $f(x) = x^2 - 2$. Compare $A(x) = |f(x)|$ and $B(x) = f(|x|)$. [4 marks]

**(a)** Find the ranges and vertical intercepts of $A$ and $B$. [2 marks]

Squaring removes the input sign in $B$, whereas $A$ reflects negative outputs upward.

**Answer:** $A$: range $[0, \infty)$, vertical intercept 2. $B = x^2 - 2$: range $[-2, \infty)$, vertical intercept $-2$.

**(b)** Find exactly where the two graphs agree. [2 marks]

Equality $|x^2 - 2| = x^2 - 2$ holds precisely where $x^2 - 2 \geq 0$.

**Answer:** $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$.

**5.** Let $f(x) = x/(x^2+1)$ and $r(x) = 1/f(x)$. [5 marks]

**(a)** Find $r$, its original domain and its range. [3 marks]

The original zero $f(0) = 0$ is excluded. The positive branch has minimum 2 at 1; odd symmetry gives the negative range.

**Answer:** $r = x + 1/x$, $x \neq 0$; range $(-\infty, -2] \cup [2, \infty)$.

**(b)** Explain why no real point of $f$ is fixed by ordinate reciprocation. [2 marks]

Solve $y = 1/y$ to get $y = \pm 1$, or use the known extrema of $f$.

**Answer:** The range of $f$ is $[-\frac{1}{2}, \frac{1}{2}]$, so it never reaches the fixed ordinates 1 or $-1$.

**6.** Let $f(x) = (x^2 + ax + b)/(x - 1)$, where $a, b$ are real. [5 marks]

**(a)** Find the condition on $a, b$ for the graph to have a hole rather than a vertical asymptote at $x = 1$. [2 marks]

The numerator must also vanish at 1; then it contains a factor $x - 1$.

**Answer:** $1 + a + b = 0$.

**(b)** If the hole is $(1, 4)$, determine $a, b$ and the simplified rule. [3 marks]

Under the condition in part (a), the quotient is $x + a + 1$. Its missing value at 1 is $a + 2 = 4$, giving $a = 2$ and $b = -3$.

**Answer:** $a = 2, b = -3$; $f = x + 3$ for $x \neq 1$.

**7.** Let $f(x) = x^3/(x^2 - 1)$. [6 marks]

**(a)** Find its asymptotes and all stationary inputs. [3 marks]

Division gives $x + x/(x^2 - 1)$; $f' = x^2(x^2 - 3)/(x^2 - 1)^2$.

**Answer:** Asymptotes $x = \pm 1$, $y = x$; stationary inputs $0, \pm\sqrt{3}$.

**(b)** Classify the stationary points and give their coordinates. [3 marks]

The derivative changes $+ \to -$ at $-\sqrt{3}$ and $- \to +$ at $\sqrt{3}$, but stays negative on both sides of zero.

**Answer:** Local maximum $(-\sqrt{3}, -3\sqrt{3}/2)$; stationary inflection $(0, 0)$; local minimum $(\sqrt{3}, 3\sqrt{3}/2)$.

**8.** Consider the family $f_a(x) = x + a/x$ on $x \neq 0$. [6 marks]

**(a)** Find the stationary points when $a > 0$, and show that none exist when $a < 0$. [3 marks]

$f_a' = 1 - a/x^2$. The equation $x^2 = a$ has real nonzero solutions only for positive $a$.

**Answer:** For $a > 0$: local maximum $(-\sqrt{a}, -2\sqrt{a})$ and local minimum $(\sqrt{a}, 2\sqrt{a})$. For $a < 0$: no stationary point.

**(b)** Compare the ranges for $a = 0$ and $a < 0$. [3 marks]

For zero parameter the rule is $x$ with its original missing input. For negative parameter each branch is strictly increasing and runs from $-\infty$ to $\infty$ between its endpoint limits.

**Answer:** $a = 0$: $\mathbb{R} \setminus \{0\}$. $a < 0$: all real numbers.

**9.** A concentration model is $c(t) = t/(t^2 + 4)$ for $t \geq 0$. [6 marks]

**(a)** Find its maximum. [4 marks]

$c'(t) = (4 - t^2)/(t^2 + 4)^2$.

The derivative changes from positive to negative at t=2.

**Answer:** Maximum 1/4 at t=2.

**(b)** State the long-term prediction and a limitation. [2 marks]

The denominator grows quadratically while the numerator grows linearly.

**Answer:** Concentration approaches zero; the idealised rule need not remain accurate indefinitely.

**10.** A student chooses both branches $3 \pm \sqrt{x - 2}$ as the inverse of $(x - 3)^2 + 2$ restricted to $x \geq 3$. [4 marks]

**(a)** Correct the inverse. [2 marks]

The original allowed input minus 3 is nonnegative.

**Answer:** $3 + \sqrt{x - 2}$ only.

**(b)** Explain why both branches cannot be used. [2 marks]

An inverse function must be single-valued.

**Answer:** The result would assign two outputs to some inverse inputs.