Functions and graph sketching - Practice test 6

10 questions | 50 marks | 75 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = \sqrt{1 - x}$ and $g(x) = x/(x - 1)$. [5 marks]

(a) Find the rule and domain of $f \circ g$. [3 marks]

$1 - x/(x - 1) = -1/(x - 1) = 1/(1 - x)$. The inner function excludes 1, and the root requires this positive expression.

Answer: $\sqrt{1/(1 - x)}$ for $x < 1$.

(b) Find the range and explain whether the composition is one-to-one. [2 marks]

As x increases from $-\infty$ to 1, $1/(1 - x)$ rises from 0 to ∞, as does its square root.

Answer: Range $(0, \infty)$; it is strictly increasing and one-to-one.

2. Let $f(x) = x^2/(x^2 + 1)$ for $x \geq 0$. [5 marks]

(a) Find its range and prove that it is one-to-one on this domain. [2 marks]

$f = 1 - 1/(x^2 + 1)$, which increases as nonnegative x increases; it starts at 0 and approaches 1.

Answer: Range $[0, 1)$; it is strictly increasing.

(b) Find f^{-1}, with its domain and range. [3 marks]

From $y = x^2/(x^2 + 1)$ obtain $x^2 = y/(1 - y)$, and choose the nonnegative root.

Answer: $f^{-1}(x) = \sqrt{x/(1 - x)}$; domain $[0, 1)$, range $[0, \infty)$.

3. Let $f(x) = (x-2)^2 + 1$ restricted to $x \leq 2$. [7 marks]

(a) Find its inverse, stating all necessary restrictions. [3 marks]

The original input lies on the left branch, so $x - 2 \leq 0$.

Answer: $f^{-1}(x) = 2 - \sqrt{x-1}$ for $x \geq 1$; inverse range $(-\infty, 2]$.

(b) Find every point common to the graph and its inverse. [4 marks]

Check $f(1) = 2$ and $f(2) = 1$, giving the first two common points. For completeness, put $u = 2 - x \geq 0$ and $v = 2 - y \geq 0$. Mutual membership gives $1 - v = u^2$ and $1 - u = v^2$, hence $(u - v)(u + v - 1) = 0$. If $u = v$, then $u^2 + u - 1 = 0$, giving $u = v = (\sqrt{5} - 1)/2$. If $u + v = 1$, then $u^2 = u$, giving $u = 0$ or $u = 1$. These yield exactly the three stated points.

Answer: $(1, 2)$, $(2, 1)$, and $(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2})$.

4. Consider $h(x) = |x^2 - 4|$. [5 marks]

(a) State the number of distinct solutions of $h(x) = c$ for $c < 0$, $c = 0$, $0 < c < 4$, $c = 4$ and $c > 4$. [3 marks]

Solve $x^2 = 4 + c$ and $x^2 = 4 - c$, counting only nonnegative right sides and avoiding duplication when $c = 0$.

Answer: Respectively $0, 2, 4, 3, 2$.

(b) Find the three intersections when $c = 4$ and identify the tangent intersection. [2 marks]

The central branch is $4 - x^2$, with horizontal tangent at zero. The outer branch gives $x^2 = 8$.

Answer: $(0, 4)$ and $(\pm 2\sqrt{2}, 4)$; the line is tangent at $(0, 4)$.

5. A nonzero ordinate y is transformed by $T(y) = 1/y$. [4 marks]

(a) Find its fixed values and show that applying T twice restores every allowed ordinate. [2 marks]

$y = 1/y$ gives $y^2 = 1$; $1/(1/y) = y$.

Answer: Fixed values ± 1; $T(T(y)) = y$ for $y \neq 0$.

(b) For $f(x) = x^2 - 1$, determine where $f(x)$ and $1/f(x)$ are both positive and where they are both negative. [2 marks]

Reciprocation preserves sign. The roots ± 1 are excluded from the reciprocal.

Answer: Both positive for $x < -1$ or $x > 1$; both negative for $-1 < x < 1$.

6. Let $F(x) = (x^2 - 2x + 1)/(x^2 - 1)$. [5 marks]

(a) Simplify and identify all holes, poles and intercepts. [3 marks]

Cancel one factor $x - 1$. The only numerator zero is still excluded from the original domain.

Answer: $F = (x - 1)/(x + 1)$ for $x \neq \pm 1$; hole $(1, 0)$; pole $x = -1$; vertical intercept $(0, -1)$ and no horizontal intercept.

(b) Find the range and explain why two values are missing. [2 marks]

The simplified rule is $1 - 2/(x + 1)$ and never equals 1. Its only zero would be the hole input 1.

Answer: $\mathbb{R} \setminus \{0, 1\}$.

7. Let $f(x) = x^2/(x - 1)$ and consider the intersections with $y = c$. [5 marks]

(a) Derive the discriminant condition for a real intersection. [3 marks]

$x^2 - cx + c = 0$ has discriminant $c^2 - 4c$. The excluded input 1 never satisfies this equation.

Answer: $c(c - 4) \geq 0$, so $c \leq 0$ or $c \geq 4$.

(b) State how many distinct intersections occur in each permitted range, including the boundary cases. [2 marks]

Positive discriminant gives two roots, zero discriminant a repeated tangent root, negative discriminant no real roots.

Answer: Two when $c < 0$ or $c > 4$; one at $c = 0$ or $c = 4$; none when $0 < c < 4$.

8. For $a > 0$, define $f_a(x) = (x^2 - a^2)/(x^2 + a^2)$. [5 marks]

(a) Find the zeros, stationary point, asymptote and range in terms of a. [3 marks]

$f_a = 1 - 2a^2/(x^2 + a^2)$; $f_a' = 4a^2x/(x^2 + a^2)^2$, with a negative-to-positive sign change at zero.

Answer: Zeros $\pm a$; global minimum $(0, -1)$; horizontal asymptote $y = 1$; range $[-1, 1)$.

(b) Describe precisely how its graph is obtained from the case $a = 1$. [2 marks]

Writing $f_a(x) = f_1(x/a)$ shows that every horizontal coordinate is multiplied by a; vertical coordinates remain the same.

Answer: A horizontal stretch by factor a.

9. A cost per unit model is $C(x) = 2 + 18/(x - 1)$ for $x > 1$. [5 marks]

(a) Find the batch size for cost 5 dollars. [3 marks]

$18/(x - 1) = 3$, so x-1=6.

Answer: x=7.

(b) Explain the horizontal asymptote in context. [2 marks]

The extra term tends to zero but stays positive.

Answer: Cost tends toward 2 dollars per unit as batch size increases.

10. A displayed curve for $x^3/(x^2 - 1)$ labels (0,0) as a maximum. [4 marks]

(a) Use the derivative to correct the label. [3 marks]

$f'(x) = x^2(x^2 - 3)/(x^2 - 1)^2$ is negative on both sides near zero.

Answer: It is a stationary inflection, not a maximum.

(b) Identify two excluded inputs the sketch must respect. [1 mark]

The denominator vanishes there.

Answer: x=-1 and x=1.