Vectors in three dimensions - practice test 5

10 questions | 60 marks | 70 minutes

Use parameters, proof and perpendicularity in connected tasks.

Equivalent correct methods earn the corresponding marks.

1. A constant force $\mathbf{F} = (2, 0, 1)$ N acts during displacement $\mathbf{d} = (-2, 1, 2)$ m. [7 marks]

(a) Calculate the work done. [2 marks]

Work is the dot product: $2(-2) + 0(1) + 1(2) = -2$.

Answer: -2 J.

(b) Find the vector component of F along the displacement direction. [3 marks]

The projection coefficient is $(\mathbf{F} \cdot \mathbf{d})/(\mathbf{d} \cdot \mathbf{d}) = -2/9$.

Multiply d by -2/9.

Answer: $(4/9, -2/9, -4/9)$ N.

(c) Explain the sign of the work in relation to the projected force. [2 marks]

The projection coefficient is negative; this indicates opposition to the chosen displacement direction.

Answer: The projected force points against the displacement, so the work is negative.

2. For arbitrary real vectors a and b, consider $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$. [7 marks]

(a) Expand and simplify this expression. [3 marks]

Distribute the dot product.

The mixed terms cancel because the dot product is commutative.

Answer: $|\mathbf{a}|^2-|\mathbf{b}|^2$

(b) Explain why the diagonals of a rhombus are perpendicular. [2 marks]

The diagonals are represented by a+b and a-b.

Use the identity and $|a|=|b|$.

Answer: For equal-length adjacent sides a,b, their sum and difference have dot product 0.

(c) Check the identity for a=(5,1,2), b=(1,0,1). [2 marks]

Direct calculation gives 28.

Answer: Both sides equal 28.

3. A parallelogram ABCD has consecutive vertices A(1,0,2), B(3,1,1), C(4,4,3). [6 marks]

(a) Find D. [2 marks]

Opposite side vectors agree, so $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$.

Combine the three coordinates.

Answer: $D = (2, 3, 4)$.

(b) Verify that its diagonals bisect one another. [2 marks]

Average A and C, then average B and D.

The two midpoint vectors agree.

Answer: Both diagonal midpoints are (5/2, 2, 5/2).

(c) Is this parallelogram a rhombus? Justify. [2 marks]

The adjacent side vectors are AB=(2,1,-1) and AD=(1,3,2).

Their squared lengths are 6 and 14, so the adjacent sides have different lengths.

Answer: No.

4. Let $\mathbf{a} = (5, 2, 3)$ and $\mathbf{b} = (1, 1, 0)$. [5 marks]

(a) Find the vector projection of a along b. [2 marks]

Projection coefficient is (7)/(2).

Multiply this scalar by b.

Answer: $(\frac{7}{2}, \frac{7}{2}, 0)$

(b) Find and verify the perpendicular component. [3 marks]

Subtract the projection from a. Dotting $(\frac{3}{2}, -\frac{3}{2}, 3)$ with b gives 0.

Answer: $(\frac{3}{2}, -\frac{3}{2}, 3)$

5. Two planes have equations $\Pi_1 : x + 2y - z = 4$ and $\Pi_2 : 2x + 4y - 2z = k$. [7 marks]

(a) Explain why their normals are parallel. [2 marks]

The coefficient vectors are (1,2,-1) and (2,4,-2).

Answer: The second normal is twice the first.

(b) For which k are the planes coincident? Describe all remaining cases. [2 marks]

Multiply every term of the first equation by 2.

Its constant becomes 8, so equality of both complete equations requires k=8.

Answer: Coincident for $k = 8$; distinct parallel planes otherwise.

(c) For k=10, find the plane separation. [3 marks]

Divide the second equation by 2 to get x+2y-z=5.

The equal-normal constants differ by 1; divide by the normal magnitude $\sqrt{6}$.

Answer: $1/\sqrt{6}$ units.

6. Consider $\ell : \mathbf{r} = (0, 0, 5) + t(1, -1, a)$ and plane $x + y + z = 3$. [4 marks]

(a) For a nonzero, find the parameter of the intersection. [2 marks]

Substitution gives $5 + at = 3$.

Divide by the stated nonzero a.

Answer: $t = -2/a$.

(b) Classify the case a=0. [2 marks]

With a=0, the left side of the plane equation stays 5.

Compare this constant with 3.

Answer: The line is parallel to and disjoint from the plane.

7. P=(0,5,2) and $\ell : \mathbf{r} = (1, 0, 0) + t(1, 1, 0)$. [5 marks]

(a) Find the parameter of the closest point Q. [2 marks]

Perpendicularity gives $(1 + t, t - 5, -2) \cdot (1, 1, 0) = 0$.

This simplifies to $2t + 1 - 5 = 0$.

Answer: $t = 2$

(b) Find Q and the shortest distance. [3 marks]

Substitute the parameter into the line.

Calculate $|Q - P| = \sqrt{22}$.

Answer: $Q = (3, 2, 0)$, distance $\sqrt{22}$.

8. Consider lines $\ell_1 = s(1, 1, 0)$ and $\ell_2 = (0, 5, 1) + t(1, 0, 1)$. [7 marks]

(a) Show that the lines are skew. [2 marks]

An intersection would require t=-1 from the z coordinate and s=-1 from the x coordinate.

Then the y coordinate would require -1=5, which is impossible.

Answer: Their directions are not parallel and the lines do not intersect.

(b) Find the closest points P and Q, and their distance. [5 marks]

The joining vector is $(t - s, 5 - s, 1 + t)$. Perpendicularity gives $t + 5 - 2s = 0$ and $2t - s + 1 = 0$.

Solve to obtain $s = 3$ and $t = 1$. Substitute in the two lines and calculate the joining-vector magnitude.

Answer: $P = (3, 3, 0)$, $Q = (1, 5, 2)$; distance $2\sqrt{3}$.

9. An augmented system has rows $[1, 1, 1 \mid 6]$, $[2, 3, 1 \mid 13]$, $[1, 0, 2 \mid 5]$. A student applies $R_2 \leftarrow R_2 - 2R_1$ but writes $[0, 1, -1 \mid 13]$. [7 marks]

(a) Identify the error and write the correct new row 2. [2 marks]

The augmented constant becomes 13-2(6)=1, not 13.

Answer: The constant must also change; the row is $[0, 1, -1 \mid 1]$.

(b) Apply $R_3 \leftarrow R_3 - R_1$ and compare the last two rows. [2 marks]

Subtract every entry of row 1 from row 3.

The two remaining equations are dependent.

Answer: $R_3 = [0, -1, 1 \mid -1]$, the negative of the corrected row 2.

(c) Describe the entire solution set and its geometry. [3 marks]

Let z=t; row 2 gives y=1+t.

Row 1 gives x=6-y-z=5-2t. There are two pivots and one free parameter.

Answer: $(x, y, z) = (5 - 2t, 1 + t, t)$ for real t; a line.

10. An augmented system has rows $[1, 2, -1 \mid 4]$, $[3, 7, -2 \mid 13]$, $[0, 0, 1 \mid 5]$. [5 marks]

(a) Apply $R_2 \leftarrow R_2 - 3R_1$. [2 marks]

Subtract 3 times every entry of row 1, including the constant.

Answer: The new row 2 is $[0, 1, 1 \mid 1]$.

(b) Find the unique solution. [3 marks]

Row 3 gives z=5; row 2 gives y=-4.

Row 1 gives x=4-2y+z=17.

Answer: $(x, y, z) = (17, -4, 5)$.