Vectors in three dimensions - practice test 6

10 questions | 73 marks | 75 minutes

Interpret data, diagnose errors and solve connected spatial problems, finishing with parameter cases.

Equivalent correct methods earn the corresponding marks.

1. A drone travels in straight segments from A to B, then B to C. Coordinates are in metres. [6 marks]

Point	x (m)	y (m)	z (m)
A	0	0	0
B	3	4	0
C	3	4	12

(a) Find the total path length. [2 marks]

AB=(3,4,0) has length 5 m.

BC=(0,0,12) has length 12 m; add the two segment lengths.

Answer: 17 m.

(b) Find the displacement from A to C and its magnitude. [2 marks]

Subtract A from C.

The squared displacement length is 9+16+144=169.

Answer: $(3, 4, 12)$ m, magnitude 13 m.

(c) Give a unit vector in the direction from A to C. [2 marks]

Divide the displacement vector by its magnitude 13.

Answer: $(3/13, 4/13, 12/13)$.

2. A vector has the form $\mathbf{a} = (1, -1, 2) + \lambda(1, 1, 0)$, where λ is real, and $|\mathbf{a}| = \sqrt{14}$. [7 marks]

(a) Verify that the two displayed component directions are perpendicular. [2 marks]

Compute 1(1)+(-1)(1)+2(0)=0.

Answer: Their dot product is zero.

(b) Find every possible value of λ. [3 marks]

The squared length is $(1+\lambda)^2 + (-1+\lambda)^2 + 4 = 6 + 2\lambda^2$.

Set this equal to 14 to obtain $\lambda^2 = 4$.

Answer: $\lambda = 2$ or $\lambda = -2$.

(c) If $\mathbf{a} \cdot (1, 1, 0) > 0$, find a. [2 marks]

The dot product is 2λ, so choose the positive root 2.

Substitute it in the original expression.

Answer: $\mathbf{a} = (3, 1, 2)$.

3. For $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (2, -1, 3)$, a student claims $\mathbf{u} \times \mathbf{v} = (5, 5, -5)$. [7 marks]

(a) Use a dot-product check to show that the claimed cross product is incorrect. [2 marks]

Compute (5,5,-5) dot (1,2,-1)=5+10+5=20.

Answer: Its dot product with u is 20, so it is not perpendicular to u.

(b) Find the correct cross product and check both perpendicularities. [3 marks]

The components are $(2(3) - (-1)(-1), (-1)(2) - 1(3), 1(-1) - 2(2))$.

The two checks give 5-10+5=0 and 10+5-15=0.

Answer: $(5, -5, -5)$.

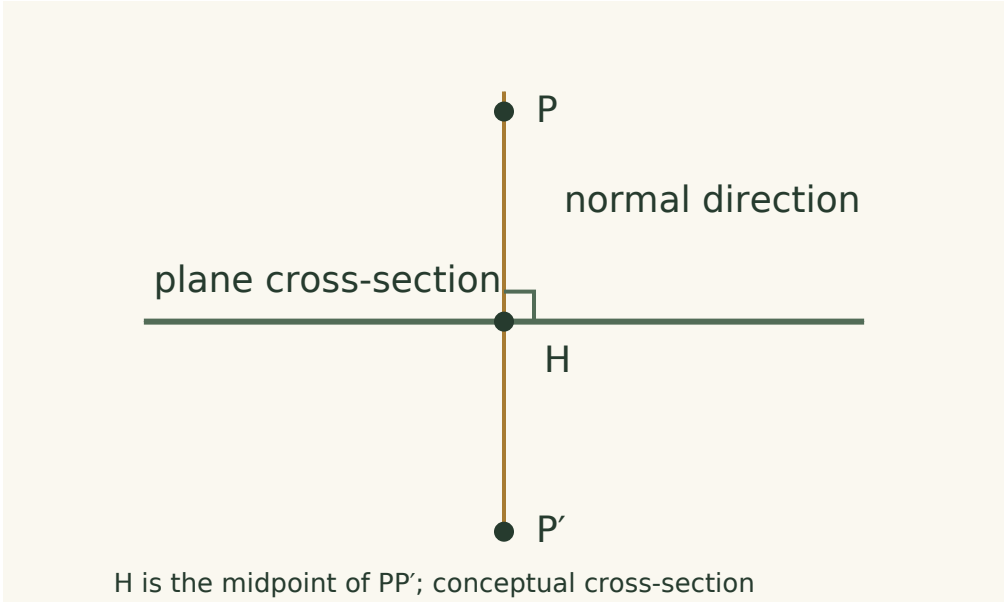
(c) Find the area of the triangle spanned by u and v. [2 marks]

The cross product has length $\sqrt{25 + 25 + 25} = 5\sqrt{3}$.

Halve this for a triangle.

Answer: $5\sqrt{3}/2$ square units.

4. Point P=(3,0,0) is reflected in the plane $x - y + z = 2$. The reflection P′ is defined so that the perpendicular foot H is the midpoint of PP′. [7 marks]



H is the midpoint of PP′; conceptual cross-section

(a) Find H. [3 marks]

A normal is n=(1,-1,1). Write H=P-mu n.

Substitute in the plane: 3-3mu=2, so mu=1/3.

Answer: $H = (8/3, 1/3, -1/3)$.

(b) Find the reflection P′. [2 marks]

The midpoint relation gives P′=2H-P.

Combine the coordinates.

Answer: $P' = (7/3, 2/3, -2/3)$.

(c) Find the length PP′ and explain why it is twice the distance of P from the plane. [2 marks]

PH has length $|3 - 2|/\sqrt{3} = 1/\sqrt{3}$.

H is the midpoint and PP′ follows the normal, so PP′=2PH.

Answer: $2/\sqrt{3}$ units.

5. Two robots move in a common coordinate system according to $P(t) = (t, 0, 0)$ and $Q(t) = (2, 3 - t, 0)$ for $0 \leq t \leq 5$, with t in seconds and coordinates in metres. [7 marks]

(a) Write their squared separation at time t. [2 marks]

Subtract the simultaneous position vectors.

Square and add the nonzero coordinate differences.

Answer: $D^2 = (t - 2)^2 + (t - 3)^2$.

(b) Find their minimum separation and the time when it occurs, without calculus. [3 marks]

Complete the square: $D^2 = 2(t - 5/2)^2 + 1/2$.

The minimum occurs at t=5/2, which is inside the stated time interval.

Answer: At $t = 5/2$ seconds, the separation is $1/\sqrt{2}$ metres.

(c) Do the robots collide? Justify. [2 marks]

A collision would require t=2 from the x coordinates and t=3 from the y coordinates at once.

Equivalently, their minimum separation is positive.

Answer: No.

6. A straight cable segment has endpoints A(1,0,0) and B(3,2,2). An infinite plane has equation $x + y + z = 10$. [7 marks]

(a) Parametrise the cable segment, stating the parameter interval. [2 marks]

Use B-A=(2,2,2) and restrict t to the endpoints.

Answer: $\mathbf{r} = (1, 0, 0) + t(2, 2, 2),\ 0 \le t \le 1$.

(b) Find where the complete line through A and B meets the plane. [2 marks]

Substitution gives 1+6t=10.

Solve for t and substitute in all three line components.

Answer: $t = 3/2$, point $(4, 3, 3)$.

(c) Does the cable segment meet the plane? Find its closest endpoint and that endpoint's distance from the plane. [3 marks]

The intersection parameter is outside [0,1]. On this interval the plane expression rises from 1 to 7, remaining below 10.

Thus B is closest; its distance is $|7 - 10|/\sqrt{3} = \sqrt{3}$.

Answer: No. B is closest, at distance $\sqrt{3}$ units.

7. Two planes are $\Pi_1 : x + y + z = 4$ and $\Pi_2 : 2x - y + z = 2$. [7 marks]

(a) Find a direction vector for their intersection. [2 marks]

Cross the normals (1,1,1) and (2,-1,1).

Answer: $(2, 1, -3)$.

(b) Give a vector equation for the intersection line. [3 marks]

Subtract the first equation from the second to get x-2y=-2.

Let y=t, then x=2t-2 and z=6-3t.

Answer: $\mathbf{r} = (-2, 0, 6) + t(2, 1, -3)$.

(c) Find the smaller angle between the planes to one decimal degree. [2 marks]

The normal dot product is 2, and the magnitudes are $\sqrt{3}$ and $\sqrt{6}$.

Thus $\cos\theta = 2/\sqrt{18} = \sqrt{2}/3$; take inverse cosine.

Answer: 61.9°.

8. Two support lines are $\ell_1 : (1, 0, 0) + s(1, 1, 1)$ and $\ell_2 : (0, 2, 0) + t(1, -1, 0)$, for real s,t. [9 marks]

(a) Find a common perpendicular direction and the shortest line separation. [3 marks]

The cross product of the line directions is (1,1,-2).

The displacement of initial points is (-1,2,0), whose dot product with this normal is 1. Divide its absolute value by $\sqrt{6}$.

Answer: A direction is $(1, 1, -2)$; distance $1/\sqrt{6}$ units.

(b) Find the closest points P on ℓ_1 and Q on ℓ_2. [4 marks]

The joining vector is $(-1 + t - s, 2 - t - s, -s)$.

Its dot products with the directions give 1-3s=0 and -3+2t=0, so s=1/3 and t=3/2.

Answer: $P = (4/3, 1/3, 1/3)$ and $Q = (3/2, 1/2, 0)$.

(c) Verify the joining vector is perpendicular to both lines. [2 marks]

Dot with (1,1,1) to get 1/6+1/6-1/3=0.

Dot with (1,-1,0) to get 1/6-1/6=0.

Answer: $\vec{PQ} = (1/6, 1/6, -1/3)$; both dot products are zero.

9. Consider the homogeneous system $x+y+z=0$, $2x-y+z=0$, $3x+2z=0$. [7 marks]

(a) Explain why the system cannot have no solution. [2 marks]

Substitute x=y=z=0 into all three equations.

Answer: The origin always satisfies every homogeneous linear equation.

(b) Find every solution and interpret the common geometry. [3 marks]

Equation 3 is the sum of equations 1 and 2.

Subtract equation 1 from equation 2 to get x=2y; set y=t, then z=-3t.

Answer: $\mathbf{r}=t(2,1,-3)$, real t; the three planes share a line.

(c) If the extra constraint x=0 is imposed, what remains? [2 marks]

Since x=2t, the extra condition gives t=0.

Answer: Only the origin $(0,0,0)$.

10. A system depends on real parameters a and b: $x+y+z=3$, $x+2y+3z=6$, $x+2y+az=b$. [9 marks]

(a) For a not equal to 3, find the unique solution in terms of a,b. [4 marks]

Subtract equation 2 from equation 3 to get $(a-3)z=b-6$.

Subtract equation 1 from equation 2 to get y+2z=3; then the first equation gives x=z.

Answer: $z=(b-6)/(a-3)$, $x=z$, $y=3-2z$.

(b) Classify and parametrise the solution set when a=3,b=6. [3 marks]

Equations 2 and 3 coincide.

Choose z=t and use the first two independent equations.

Answer: Infinitely many solutions $(t,3-2t,t)$, real t; a line.

(c) Classify the case a=3,b not equal to 6 and justify it. [2 marks]

Equations 2 and 3 then have identical left sides and different constants, which is a contradiction.

Answer: No solution.