Integration techniques and applications - practice test 5

10 questions | 58 marks | 75 minutes

Reasoning and applications across exact integrals, numerical approximation, changing regions and motion.

Equivalent correct methods earn the corresponding marks.

1. For all real x, $F'(x) = (2x + 3)/(x^2 + 3x + 5)$ and F(0)=0. [5 marks]

(a) Find F and justify that it is defined on the whole real line. [3 marks]

Use the derivative-of-denominator pattern; the constant is $-\ln 5$.

$x^2 + 3x + 5 = (x + 3/2)^2 + 11/4 > 0$.

Answer: $F(x) = \ln((x^2 + 3x + 5)/5)$.

(b) Find F(2)-F(0). [2 marks]

The logarithm ratio at 2 is $15/5 = 3$.

Answer: $\ln 3$.

2. Let $I = \int_0^1 (1+x^2)^{-1}\,dx$ and $J = \int_0^1 (1+4x^2)^{-1}\,dx$. [5 marks]

(a) Evaluate I and J exactly. [3 marks]

For I use $\arctan x$. For J substitute $u = 2x$, including $dx = du/2$.

Answer: $I = \pi/4$; $J = \frac{1}{2}\arctan 2$.

(b) Without using decimal approximations, explain which is larger. [2 marks]

For every $0 < x \le 1$, $1 + x^2 < 1 + 4x^2$, so the first positive reciprocal is larger.

Integrating this strict inequality over a positive-length interval preserves it.

Answer: $I > J$.

3. For $x > 0$, let $f(x) = (x^2 + x + 1)/[x(x+1)^2]$. [6 marks]

(a) Find A, B and C in $f = A/x + B/(x+1) + C/(x+1)^2$. [3 marks]

Clear denominators: $x^2 + x + 1 = A(x+1)^2 + Bx(x+1) + Cx$.

The constant gives $A = 1$; then the quadratic gives $B = 0$ and the linear gives $C = -1$.

Answer: $A = 1, B = 0, C = -1$.

(b) Evaluate the integral of f from 1 to 2. [3 marks]

An antiderivative is $\ln x + 1/(x+1)$.

The endpoint difference is $\ln 2 + 1/3 - 1/2$.

Answer: $\ln 2 - 1/6$.

4. Let $I = \int_0^{\pi/2} e^x \sin x \, dx$ and $J = \int_0^{\pi/2} e^x \cos x \, dx$. [6 marks]

(a) Use integration by parts to show that I+J equals $e^{\pi/2}$. [3 marks]

Choose $u = \sin x$, $dv = e^x \, dx$.

The endpoint product $[e^x \sin x]_0^{\pi/2}$ equals $e^{\pi/2}$.

Answer: $I = e^{\pi/2} - J$.

(b) Use parts on J, then solve for I. [3 marks]

With $u = \cos x$, $dv = e^x \, dx$, obtain $J = [e^x \cos x]_0^{\pi/2} + I = -1 + I$.

Substitute into $I + J = e^{\pi/2}$ and solve $2I - 1 = e^{\pi/2}$.

Answer: $I = (e^{\pi/2} + 1)/2$.

5. The curve is $y = x^3 - x$ on $-2 \le x \le 2$. [6 marks]

(a) Find all zeros in the interval and the signed integral. [2 marks]

Factor $x(x-1)(x+1)$.

The integrand is odd and the limits are symmetric.

Answer: Zeros $-1, 0, 1$; signed integral 0.

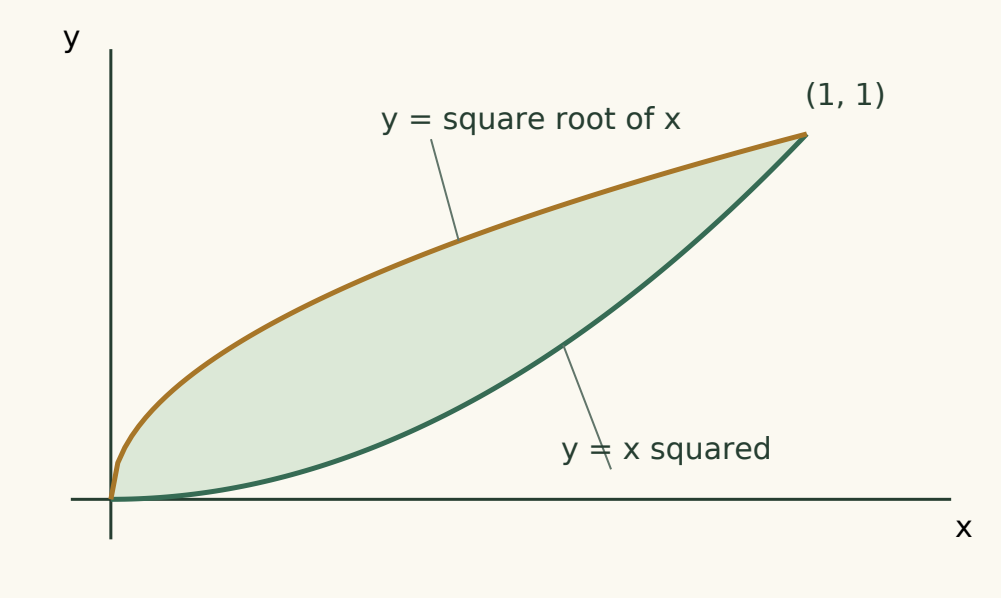
(b) Find the total geometric area between the curve and the axis. [4 marks]

Use symmetry and split the positive half at 1: $A = 2[\int_0^1 (x - x^3) \, dx + \int_1^2 (x^3 - x) \, dx]$.

The two bracketed areas are 1/4 and 9/4, so $A = 5$.

Answer: 5 square units.

6. A region lies between $y = \sqrt{x}$ and $y = x^2$ in the first quadrant. [9 marks]



(a) Find the intersection points and the bounded area. [3 marks]

For $x \geq 0$, $\sqrt{x} = x^2$ gives $x = 0$ or $x = 1$.

On $(0, 1)$, the square root is above the square. Integrate $x^{1/2} - x^2$ to get $2/3 - 1/3$.

Answer: $(0, 0), (1, 1)$; area $1/3$ square unit.

(b) Find its volume when rotated about the x axis. [3 marks]

Outer radius is $\sqrt{x}$ and inner radius is x^2.

$V_x = \pi \int_0^1 (x - x^4)\, dx = \pi(1/2 - 1/5)$.

Answer: $3\pi/10$ cubic unit.

(c) Find its volume when rotated about the y axis, using horizontal slices. [3 marks]

The boundaries become $x = y^2$ and $x = \sqrt{y}$ for $0 \leq y \leq 1$.

$V_y = \pi \int_0^1 (y - y^4)\, dy = 3\pi/10$.

Answer: $3\pi/10$ cubic unit.

7. A tank has horizontal cross-sectional area $A(h) = \pi(4 - h^2)$ m^2 at height h metres, for $0 \le h \le 2$. [5 marks]

(a) Find the volume below height h and the full capacity. [3 marks]

Integrate the cross-sectional area from 0 to h.

Evaluate the resulting volume function at h=2.

Answer: $V(h) = \pi(4h - h^3/3)$ m^3; capacity $16\pi/3$ m^3.

(b) At half the full height, what fraction of the full capacity is filled? [2 marks]

$V(1) = 11\pi/3$.

Divide by $V(2) = 16\pi/3$. Half height does not mean half volume because cross-sectional area varies.

Answer: 11/16.

8. The table gives values of $f(x) = e^{-x^2}$. Use these values as supplied. [5 marks]

x	f(x)
0	1.0000
0.5	0.7788
1	0.3679

(a) Use the trapezium rule with two equal strips to estimate the integral from 0 to 1. [3 marks]

The strip width is $h = 0.5$.

$T = (0.5/2)[1 + 2(0.7788) + 0.3679] = 0.731375$.

Answer: 0.731375.

(b) A more accurate numerical value is 0.746824. Find the signed error T minus this value and the absolute error. [2 marks]

Subtract 0.746824 from the trapezium result.

The estimate is below the supplied more accurate value.

Answer: Signed error -0.015449; absolute error 0.015449.

9. A periodic temperature model is $T(t) = 3 + 2\cos t + \sin^2 t$ degrees C for $0 \le t \le 2\pi$ hours. [5 marks]

(a) Find its exact average temperature over this interval. [3 marks]

The constant contributes 6π, cosine integrates to zero, and sine squared contributes π.

Divide total 7π degree-hours by interval length 2π hours.

Answer: 7/2 degrees C.

(b) Explain why the average is above 3 degrees C despite the cosine term averaging to zero. [2 marks]

A zero-mean cosine does not cancel a separate non-negative sine-squared contribution.

Answer: The sine-squared term has positive average 1/2.

10. A particle moves on a straight line with velocity $v(t) = 3t^2 - 6t$ m/s for $0 \le t \le 3$. Its initial position is 2 m. [6 marks]

(a) Find its final position and average signed velocity. [3 marks]

The displacement is $[t^3 - 3t^2]_0^3 = 0$.

Add the initial position; divide displacement by 3 for average velocity.

Answer: Final position 2 m; average signed velocity 0 m/s.

(b) Find its total distance and average speed. [3 marks]

Velocity changes sign at t=2. The signed changes over $[0, 2]$ and $[2, 3]$ are -4 m and 4 m.

Add their magnitudes, then divide by elapsed time.

Answer: Distance 8 m; average speed 8/3 m/s.