Integration techniques and applications - practice test 6

10 questions | 64 marks | 75 minutes

Challenging combined-method problems, parameter reasoning, singularities and geometric models.

Equivalent correct methods earn the corresponding marks.

1. For $x > 0$, consider $\int x^n \ln x \, dx$, where n is real. [7 marks]

(a) For $n \neq -1$, derive a general antiderivative using parts. [4 marks]

Take $u = \ln x$ and $dv = x^n \, dx$, so $v = x^{n+1}/(n+1)$.

The remaining integral is $\frac{1}{n+1}\int x^n \, dx$.

Answer: $\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$.

(b) Handle $n = -1$ separately and evaluate $\int_1^{e^2} (\ln x)/x \, dx$. [3 marks]

Use $u = \ln x$, $du = dx/x$.

The transformed limits are 0 and 2; integrate u from 0 to 2.

Answer: Antiderivative $(\ln x)^2/2 + C$; definite value 2.

2. Let $I = \int_0^1 x^3/(1+x^2) \, dx$. [6 marks]

(a) Use polynomial division to find an antiderivative. [3 marks]

Write $x^3/(1+x^2) = x - x/(1+x^2)$.

Integrate the first term directly and the second by $u = 1 + x^2$.

Answer: $x^2/2 - \frac{1}{2}\ln(1+x^2) + C$.

(b) Evaluate I and show how the single substitution u=1+x squared gives the same result. [3 marks]

Under the substitution, $x^3 dx = (u-1)du/2$ and the limits are 1 and 2.

$I = \frac{1}{2}\int_1^2 (1 - 1/u) \, du = (1 - \ln 2)/2$.

Answer: $I = (1 - \ln 2)/2$.

3. Consider $I = \int_0^{\pi/2} \cos^4 x\,dx$. [5 marks]

(a) Derive an identity expressing cosine to the fourth as a sum of cosines. [3 marks]

Square $\cos^2 x = (1 + \cos 2x)/2$.

Reduce $\cos^2(2x)$ using $(1 + \cos 4x)/2$ and collect terms.

Answer: $\cos^4 x = (3 + 4\cos 2x + \cos 4x)/8$.

(b) Evaluate I exactly. [2 marks]

Integrate to $3x/8 + \sin 2x/4 + \sin 4x/32$.

Both sine terms vanish at both endpoints.

Answer: $3\pi/16$.

4. Consider $f(x) = (x^2 + 2)/(x^3 - x)$. [8 marks]

(a) Find its partial-fraction decomposition. [3 marks]

Factor the denominator as $x(x - 1)(x + 1)$ and clear denominators.

At x=0 the coefficient of 1/x is -2; at x=1 and -1 the other coefficients are both 3/2.

Answer: $f(x) = -2/x + 3/[2(x - 1)] + 3/[2(x + 1)]$.

(b) Evaluate its integral from 2 to 3 exactly. [3 marks]

An antiderivative on this interval is $-2\ln x + \frac{3}{2}\ln(x - 1) + \frac{3}{2}\ln(x + 1)$.

Combine endpoint differences in the two latter logs to obtain $\frac{3}{2}\ln[(3^2 - 1)/(2^2 - 1)]$.

Answer: $-2\ln(3/2) + \frac{3}{2}\ln(8/3)$.

(c) Explain why the same ordinary endpoint calculation cannot be used from 1 to 3. [2 marks]

The integrand is undefined there; the ordinary continuity hypothesis fails and its logarithmic antiderivative has no finite endpoint value.

Answer: There is a pole at the lower endpoint x=1.

5. Seek an antiderivative of $e^{2x}\cos x$ in the form $F(x) = e^{2x}(A\cos x + B\sin x)$. [6 marks]

(a) Differentiate the proposed form and find A and B. [3 marks]

The derivative is $e^{2x}[(2A + B)\cos x + (2B - A)\sin x]$.

Solve $2A + B = 1$, $2B - A = 0$.

Answer: $A = 2/5, B = 1/5$.

(b) Evaluate $\int_0^{\pi} e^{2x}\cos x\,dx$. [3 marks]

The antiderivative values are $-2e^{2\pi}/5$ at pi and 2/5 at zero.

Subtract the complete lower value.

Answer: $-2(e^{2\pi} + 1)/5$.

6. Let $I = \int_0^{\ln 2} e^x/\sqrt{1 + e^x}\,dx$. [5 marks]

(a) Choose a substitution, change the bounds and evaluate I. [3 marks]

Let $u = 1 + e^x$, so $du = e^x\,dx$ and the bounds become 2 and 3.
$I = \int_2^3 u^{-1/2}\,du = [2\sqrt{u}]_2^3$.

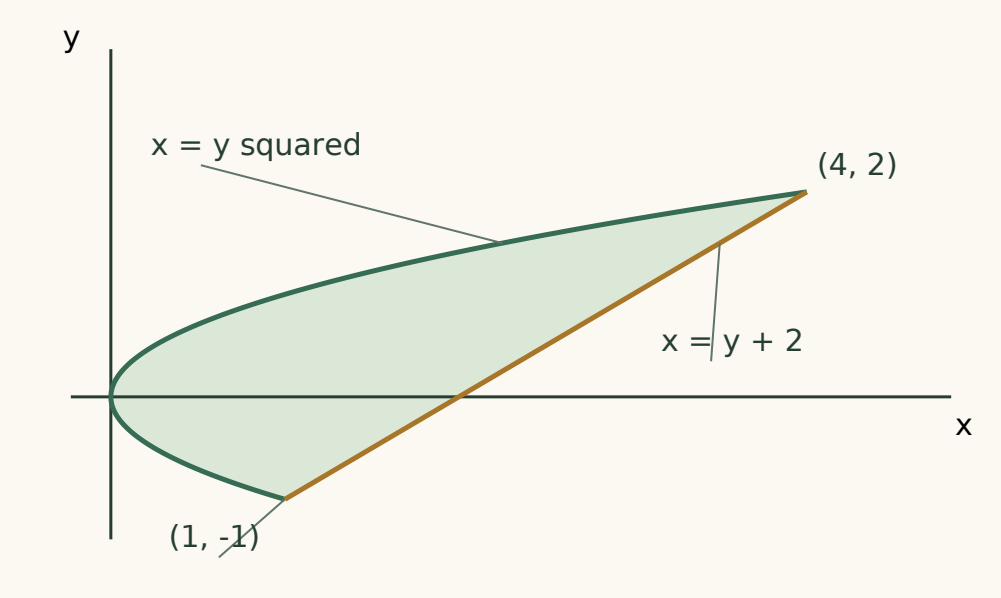
Answer: $I = 2(\sqrt{3} - \sqrt{2})$.

(b) Find the average value of the integrand on the original interval. [2 marks]

Divide the integral by the positive interval length $\ln 2$.

Answer: $2(\sqrt{3} - \sqrt{2})/\ln 2$.

7. The region bounded by $x = y^2$ and $x = y + 2$ is rotated about the y axis. [7 marks]



(a) Find the two intersection y values and the plane area. [3 marks]

$y^2 = y + 2$ gives $(y - 2)(y + 1) = 0$. Between the roots, $y + 2$ is the right boundary.

$A = \int_{-1}^{2}(y + 2 - y^2)\,dy = 9/2$.

Answer: $y = -1, 2$; area 9/2 square units.

(b) Find the exact solid volume using washers. [4 marks]

Both radii are non-negative on this interval: outer $y + 2$ and inner y^2.

$V = \pi\int_{-1}^{2}[(y + 2)^2 - y^4]\,dy = 72\pi/5$.

Answer: $72\pi/5$ cubic units.

8. The region $0 \le x \le \pi$, $0 \le y \le \sin x$ is rotated about the x axis. [6 marks]

(a) Find the total volume. [3 marks]

The disk radius is $\sin x$, so $V = \pi\int_0^{\pi} \sin^2 x\, dx$.

Use the half-angle identity; the integral is $\pi/2$.

Answer: $\pi^2/2$ cubic units.

(b) Find the volume from x=0 to a, where $0 \le a \le \pi$, and show that the plane x=pi/2 divides the solid into equal volumes. [3 marks]

Integrate the squared radius with upper limit a.

At $a = \pi/2$, the sine term is zero and the result is half the total volume.

Answer: $V(a) = \pi(a/2 - \sin 2a/4)$; $V(\pi/2) = \pi^2/4$.

9. A tank starts with 30 L. During $0 \le t \le 4$ minutes, inflow is $6t(4 - t)$ L/min and outflow is a constant 8 L/min. [7 marks]

(a) Find the total inflow, total outflow and final amount. [3 marks]

$\int_0^4 (24t - 6t^2)\, dt = [12t^2 - 2t^3]_0^4 = 64$.

Outflow is $8(4) = 32$ L; add the net change 32 L to the initial 30 L.

Answer: Inflow 64 L; outflow 32 L; final amount 62 L.

(b) Find an amount function V(t), including its initial amount. [2 marks]

Integrate the net rate and impose $V(0) = 30$.

Answer: $V(t) = 30 + 12t^2 - 2t^3 - 8t$ L.

(c) Explain why the amount initially decreases even though the final amount exceeds 30 L. [2 marks]

The initial net rate is -8 L/min. Later positive accumulation can exceed this early loss.

Answer: At t=0, the inflow is zero while the outflow is 8 L/min.

10. For $a > 0$, the region under $y = ax(1 - x)$ on $0 \le x \le 1$ has area A. Its rotation about the x axis has volume V. [7 marks]

(a) Find A and V in terms of a. [4 marks]

$A = a\int_0^1 (x - x^2)\,dx = a/6$.

$V = \pi a^2 \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \pi a^2(1/3 - 1/2 + 1/5)$.

Answer: $A = a/6$; $V = \pi a^2/30$.

(b) Eliminate a to express V in terms of A, then find V when A=2. [3 marks]

$a = 6A$, so substitute into the volume expression.

The quadratic dependence means doubling the area by vertical scaling quadruples this rotational volume.

Answer: $V = 6\pi A^2/5$; for A=2, $V = 24\pi/5$.