Rates of change and differential equations - Test 6

10 questions | 63 marks | 75 minutes

Combine methods and critique common modelling errors.

Equivalent correct methods earn the corresponding marks.

1. The curve $x^2y + xy^2 = 6$ passes through (1, 2). [6 marks]

(a) Find dy/dx. [3 marks]

Differentiate to $2xy + x^2y' + y^2 + 2xyy' = 0$.

Answer: $y' = -(2xy + y^2)/(x^2 + 2xy)$

(b) Find the tangent at the stated point. [3 marks]

At (1, 2), the numerator is 8 and denominator 5.

Use point-gradient form.

Answer: $y - 2 = -\frac{8}{5}(x - 1)$

2. A rectangle has side x and perimeter 30 cm. Its x side increases at 1 cm/s. At x=4 cm: [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

The other side is 15-x.

Answer: $A = x(15 - x)$

(b) Find the area rate in square centimetres per second. [3 marks]

$dA/dt = (15 - 2x)dx/dt = (15 - 8)(1)$.

The positive sign means that the area is increasing.

Answer: $dA/dt = 7$ cm^2/s

3. Two proposed solutions are $f(x) = 2e^x$ and $g(x) = e^{2x}$. The initial-value problem is $y' = 2y$, $y(0) = 2$. [6 marks]

(a) Explain which requirement each proposed function fails. [3 marks]

$f(0) = 2$, but $f' = 2e^x$ whereas $2f = 4e^x$.

$g' = 2e^{2x} = 2g$, but $g(0) = 1$.

Answer: f meets the initial condition but fails the differential equation. g meets the differential equation but fails the initial condition.

(b) Find the correct solution and verify both requirements. [3 marks]

The equation gives $y = Ae^{2x}$ and the initial value sets $A = 2$.

Then $y' = 4e^{2x} = 2y$ and $y(0) = 2$.

Answer: $y = 2e^{2x}$.

4. The equation is $y' = 1 - y^2$. A numerical sketch begins at $y(0) = 0$. [6 marks]

(a) Identify and classify the constant equilibria using the slope signs. [3 marks]

The slope vanishes at $y = \pm 1$.

It is negative below -1, positive between -1 and 1, and negative above 1.

Answer: $y = -1$ is unstable and $y = 1$ is stable.

(b) A sketch crosses y=1 with positive slope. Explain the error, and verify the proposed exact solution $y = (e^{2x} - 1)/(e^{2x} + 1)$. [3 marks]

At $y = 1$ the equation requires slope zero, so a positive-slope crossing is inconsistent.

Quotient differentiation gives $y' = 4e^{2x}/(e^{2x} + 1)^2$.

Also $1 - y^2 = 4e^{2x}/(e^{2x} + 1)^2$, and $y(0) = 0$.

Answer: The proposed solution satisfies the equation and initial condition, and lies strictly between -1 and 1 for finite x.

5. A particle has acceleration $a(t) = 6t - 6$ m/s^2, $v(0) = 0$ and $x(0) = 5$ m for $0 \le t \le 3$ s. [6 marks]

(a) Find its velocity and position. [3 marks]

Integrating acceleration gives $v = 3t^2 - 6t + C_1$; $v(0) = 0$ sets $C_1 = 0$.

Integrate again and apply $x(0) = 5$.

Answer: $v = 3t^2 - 6t$ m/s; $x = t^3 - 3t^2 + 5$ m.

(b) Find its displacement and total distance during the three seconds. [3 marks]

The velocity changes sign at $t = 2$ in the interior.

$x(0) = 5$, $x(2) = 1$ and $x(3) = 5$.

Displacement is $5 - 5 = 0$; distance is $|1 - 5| + |5 - 1| = 8$.

Answer: Displacement 0 m; total distance 8 m.

6. Consider $y' = y/(x + 1)$, $y(0)$=-2, $x > -1$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

Separate $dy/y = dx/(x + 1)$, giving $\ln|y| = \ln(x + 1) + C$.

Retain a signed multiplier: $y = A(x + 1)$. The condition gives $A = -2$; the original equation requires $x \neq -1$.

Answer: $y = -2(x + 1)$, on $x > -1$.

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

A zero function solves the equation on its permitted x domain.

Answer: $y = 0$

7. A reservoir contains V litres and satisfies $V' = 4 - 0.1V$, $V(0) = 10$, with t in minutes. The inflow is 4 L/min and the outflow rate is 0.1 V L/min. [6 marks]

(a) Find the amount model. [3 marks]

Shift to V-40, which decays exponentially.

Answer: $V = 40 - 30e^{-0.1t}$

(b) Find the long-term amount and the initial rate. [3 marks]

As the exponential vanishes the limit is 40; 4-0.1(10)=3.

Answer: Long-term amount 40 L; initial rate 3 L/min.

8. A population satisfies $P' = 0.2P(1 - P/900)$ with $P(0) = 600$. Here t is in days, P is in individuals, and t is at least zero. [9 marks]

(a) Find $P(t)$ and the carrying capacity. [3 marks]

$A = (900 - 600)/600 = 0.5$.

Answer: $P = 900/(1 + 0.5e^{-0.2t})$; carrying capacity 900.

(b) Find the global largest possible positive growth rate and when this trajectory reaches its associated amount. [3 marks]

The quadratic rate has its vertex at P=K/2, with value rK/4.

The maximum positive growth along this future trajectory is initially; K/2 occurred at a negative model time.

Answer: The global rate maximum is 45 individuals/day at P=450. This trajectory never reaches P=450 for nonnegative t.

(c) For this initial population, find the greatest growth rate for t at least zero. [3 marks]

The trajectory starts at P=600, already above K/2=450.

For P above K/2 the logistic rate decreases as P increases.

The initial rate is $0.2(600)(1 - 600/900) = 40$.

Answer: 40 individuals/day at t=0.

9. A displacement model is $x = 2\cos 3t - 2\sin 3t$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Find or verify the displacement and its derivative relation. [3 marks]

Differentiate both terms twice.

Answer: $v = -6\sin 3t - 6\cos 3t$, $a = -9x$

(b) Find the amplitude and maximum speed. [3 marks]

Use amplitude square root of 4+4, then multiply by 3.

Answer: Amplitude $2\sqrt{2}$; maximum speed $6\sqrt{2}$.

10. For $y' = x - y$, a student calls $y = x$ an equilibrium because its field slopes vanish. [6 marks]

(a) Identify and correct the error. [3 marks]

Along $y = x$, the derivative of the proposed function is 1, but the differential equation requires $x - x = 0$.

Zero slopes along a nonhorizontal line do not make that line a solution.

Answer: This is the zero-slope line, not an equilibrium solution.

(b) Prove that no constant equilibrium exists. [3 marks]

If $y = c$ for a constant c, then $y' = 0$.

The right side is $x - c$, which cannot be zero for every x in an interval.

Answer: No constant equilibrium exists.