Vector calculus - Test 1

10 questions | 60 marks | 75 minutes

Parametric foundations and motion.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = t + 1, y = t^2$, $0 \le t \le 2$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$t = x - 1$, so $y = (x - 1)^2$.

As $0 \le t \le 2$, the traced x interval is $1 \le x \le 3$.

Answer: $y = (x - 1)^2$, $1 \le x \le 3$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

$\mathbf{r}(0) = (1, 0)$ and $\mathbf{r}(2) = (3, 4)$.

$dx/dt = 1 > 0$, so the curve is traced once from left to right.

Answer: From (1, 0) to (3, 4), left to right.

2. In metres at time t seconds, A is at $(1 + 2t, 3 - t)$ and B at $(7 - t, t/2)$, for $t \ge 0$. [6 marks]

(a) Find whether and when they collide. [3 marks]

$1 + 2t = 7 - t$ gives $t = 2$.

At that time, both y coordinates equal 1, so the collision point is $(5, 1)$.

Answer: t=2 s at (5, 1) m.

(b) Find their speeds. [3 marks]

$\mathbf{v}_A = (2, -1)$ and $\mathbf{v}_B = (-1, 1/2)$.

Their magnitudes are $\sqrt{5}$ and $\sqrt{1 + 1/4} = \sqrt{5}/2$ metres per second.

Answer: A:$\sqrt{5}$ m/s; B:$\sqrt{5}/2$ m/s.

3. $\mathbf{r} = (t, 6t - t^2)$, $0 \leq t \leq 6$. [6 marks]

(a) Find the greatest height and the velocity there. [3 marks]

$dy/dt = 6 - 2t = 0$ gives $t = 3$; $d^2y/dt^2 = -2 < 0$.

$y(3) = 9$ and $\mathbf{v}(3) = (1, 0)$.

Answer: Maximum y=9 at t=3; velocity (1, 0).

(b) Can the particle ever stop? Explain. [3 marks]

$\mathbf{v} = (1, 6 - 2t)$.

The first component is always 1, so the velocity vector is never zero.

Answer: It never stops because x velocity is 1.

4. A quadratic Bézier path has controls (0, 0), (2, 4), (6, 0). Use the standard Bézier parameter interval $0 \leq t \leq 1$. [6 marks]

(a) Find B (1/2). [3 marks]

$\mathbf{B}(1/2) = \frac{1}{4}(0, 0) + \frac{1}{2}(2, 4) + \frac{1}{4}(6, 0) = (2.5, 2)$.

Answer: $(2.5, 2)$

(b) Find both endpoint derivative vectors. [3 marks]

$\mathbf{B}'(0) = 2[(2, 4) - (0, 0)] = (4, 8)$.

$\mathbf{B}'(1) = 2[(6, 0) - (2, 4)] = (8, -8)$.

Answer: $(4, 8)$ and (8, -8).

5. $\mathbf{v} = (2t, 3)$, $\mathbf{r}(0) = (1, -2)$. [6 marks]

(a) Find the position function. [3 marks]

Integration gives $\mathbf{r} = (t^2 + C_1, 3t + C_2)$.

The initial vector gives $C_1 = 1$ and $C_2 = -2$.

Answer: $\mathbf{r} = (t^2 + 1, 3t - 2)$

(b) Find displacement from 0 to 2. [3 marks]

$\mathbf{r}(2) = (5, 4)$ and $\mathbf{r}(0) = (1, -2)$.

The displacement is their difference, $(4, 6)$.

Answer: Displacement from 0 to 2 is (4, 6).

6. $\mathbf{r} = (t^2, 2t)$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (2t, 2)$ and $\mathbf{a} = (2, 0)$.

Answer: $\mathbf{v} = (2t, 2), \mathbf{a} = (2, 0)$

(b) Find speed and speed rate at 1. [3 marks]

At 1, speed is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

$d|\mathbf{v}|/dt = (\mathbf{v} \cdot \mathbf{a})/|\mathbf{v}| = 4/(2\sqrt{2}) = \sqrt{2}$.

Answer: Speed at 1 is $2\sqrt{2}$;speed rate at 1 is $\sqrt{2}$.

7. For $x = t^2, y = t^3$, t=1. [6 marks]

(a) Find the point and tangent gradient at $t = 1$. [3 marks]

At 1 the point is $(1, 1)$ and the derivative vector is $(2, 3)$.

The tangent gradient is 3/2.

Answer: Point (1, 1); gradient 3/2.

(b) Give the tangent line. [3 marks]

The point-gradient equation is $y - 1 = \frac{3}{2}(x - 1)$.

Answer: $y - 1 = \frac{3}{2}(x - 1)$.

8. $\mathbf{r} = (2\cos 3t, 2\sin 3t)$. [6 marks]

(a) Describe the locus and find its period. [3 marks]

$x^2 + y^2 = 4$, so the circle has centre $(0, 0)$ and radius 2.

The angular speed is 3, giving period $2\pi/3$.

Answer: Circle centre (0, 0), radius 2; period $2\pi/3$.

(b) Find speed and acceleration magnitude. [3 marks]

$|\mathbf{v}| = 2(3) = 6$ and $|\mathbf{a}| = 2(3^2) = 18$.

Answer: Speed 6; acceleration magnitude 18.

9. $\mathbf{r} = (3t, 4t), 0 \le t \le 2$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{r}' = (3, 4)$, so $|\mathbf{r}'| = 5$.

$L = \int_0^2 5\,dt$.

Answer: $L = \int_0^2 5dt$

(b) Evaluate the exact curve length. [3 marks]

$L = [5t]_0^2 = 10$.

Answer: $L = 10$

10. A camera follows $\mathbf{r} = (t, 4t - t^2)$ metres, $0 \le t \le 4$. [6 marks]

(a) Find initial/final positions and the greatest height. [3 marks]

The endpoints are $(0, 0)$ and $(4, 0)$.

$dy/dt = 4 - 2t = 0$ at 2; the maximum height is $y(2) = 4$ m.

Answer: (0, 0) and (4, 0); greatest height 4 at 2.

(b) Find velocity at the greatest height and explain why the camera is still moving. [3 marks]

$\mathbf{v} = (1, 4 - 2t)$, so $\mathbf{v}(2) = (1, 0)$ m/s.

The horizontal component is nonzero, so the camera keeps moving.

Answer: $(1, 0)$ m/s; horizontal velocity is nonzero.