Vector calculus - Test 2

10 questions | 60 marks | 75 minutes

Trace restrictions, branches and repeated paths.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = t^2, y = 2t, -1 \le t \le 1$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$t = y/2$ gives $x = y^2/4$.

The interval gives $-2 \le y \le 2$.

Answer: $x = y^2/4, -2 \le y \le 2$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

At $t = -1, 0, 1$, the points are $(1, -2), (0, 0), (1, 2)$.

Because y increases strictly, no point is retraced.

Answer: From (1, -2) through (0, 0) to (1, 2).

2. A is at (t, t) and B at $(2 - t, 3 - t)$, for $t \ge 0$. [6 marks]

(a) Show that there is no collision. [3 marks]

Equal x coordinates require $t = 1$.

The y coordinates are then 1 and 2, so there is no collision.

Answer: Equal x requires t=1, but y values are 1 and 2.

(b) Find the distance between them when their x coordinates agree. [3 marks]

At 1 the positions are $(1, 1)$ and $(1, 2)$.

The separation vector is $(0, 1)$ and its magnitude is 1.

Answer: 1 unit.

3. $\mathbf{r} = (t^2, t^3 - t)$, $-1 \le t \le 1$. [6 marks]

(a) Find and verify a repeated point. [3 marks]

$\mathbf{r}(-1) = \mathbf{r}(1) = (1, 0)$.

Answer: Both t=-1 and 1 give (1, 0).

(b) Find velocity at t=0 and decide whether it is at rest. [3 marks]

$\mathbf{v} = (2t, 3t^2 - 1)$.

$\mathbf{v}(0) = (0, -1) \neq \mathbf{0}$, so the particle is moving.

Answer: At t=0 velocity (0, -1), so the origin is not a rest point.

4. A cubic Bézier path has controls (0, 0), (1, 2), (3, 2), (4, 0). Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Find the start and end positions. [3 marks]

At 0 only the first Bézier weight remains; at 1 only the last remains.

The endpoints are $(0, 0)$ and $(4, 0)$.

Answer: (0, 0) and (4, 0).

(b) Find both endpoint derivatives and explain the tangent directions. [3 marks]

$\mathbf{B}'(0) = 3(1, 2) = (3, 6)$.

$\mathbf{B}'(1) = 3[(4, 0) - (3, 2)] = (3, -6)$, giving the stated directions.

Answer: (3, 6) and (3, -6); up-right initially, down-right finally.

5. $\mathbf{a} = (6t, 2), \mathbf{v}(0) = (1, -1), \mathbf{r}(0) = (0, 3)$. [6 marks]

(a) Find the position function. [3 marks]

$\mathbf{v} = (3t^2 + C_1, 2t + C_2) = (3t^2 + 1, 2t - 1)$.

Integrating again and applying position at 0 gives $\mathbf{r} = (t^3 + t, t^2 - t + 3)$.

Answer: $\mathbf{v} = (3t^2 + 1, 2t - 1)$; $\mathbf{r} = (t^3 + t, t^2 - t + 3)$

(b) Find position and velocity at t=1. [3 marks]

$\mathbf{r}(1) = (2, 3)$ and $\mathbf{v}(1) = (4, 1)$.

Answer: At t=1, position (2, 3), velocity (4, 1).

6. $\mathbf{r} = (3\cos t, 3\sin t)$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (-3\sin t, 3\cos t)$ and $\mathbf{a} = (-3\cos t, -3\sin t)$.

Answer: $\mathbf{v} = (-3\sin t, 3\cos t)$,$\mathbf{a} = (-3\cos t, -3\sin t)$

(b) Find speed and acceleration magnitude and explain whether speed changes. [3 marks]

$|\mathbf{v}|^2 = 9(\sin^2 t + \cos^2 t) = 9$, and likewise $|\mathbf{a}| = 3$.

$\mathbf{v} \cdot \mathbf{a} = 0$, so speed does not change.

Answer: Speed 3, constant; acceleration magnitude 3.

7. For $x = t^2, y = t$, t=0. [6 marks]

(a) Find the point and derivative vector at $t = 0$. What does the vector show about the tangent? [3 marks]

At 0 the point is $(0, 0)$ and the derivative vector is $(0, 1)$.

The tangent is vertical.

Answer: Point (0, 0); vertical direction (0, 1).

(b) Give the tangent line. [3 marks]

The vertical line through $(0, 0)$ is $x = 0$.

Answer: $x = 0$.

8. $\mathbf{r} = (3\cos t, 2\sin t)$. [6 marks]

(a) Describe the locus and find its period. [3 marks]

$x^2/9 + y^2/4 = 1$, an ellipse.

Both components repeat after 2π.

Answer: Ellipse $x^2/9 + y^2/4 = 1$; period 2π.

(b) Compare speeds at 0 and $\pi/2$; is speed constant? [3 marks]

$\mathbf{v} = (-3\sin t, 2\cos t)$.

The speeds at 0 and $\pi/2$ are 2 and 3, so speed is not constant.

Answer: Speed 2 at 0 and 3 at $\pi/2$; not constant.

9. $\mathbf{r} = (t^2, \frac{2}{3}t^3), 0 \le t \le 1$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{r}' = (2t, 2t^2)$; on this interval, speed is $2t\sqrt{1+t^2}$.

$L = \int_0^1 2t\sqrt{1+t^2}\,dt$.

Answer: $L = \int_0^1 2t\sqrt{1+t^2}\,dt$

(b) Evaluate the exact curve length. [3 marks]

Let $u = 1 + t^2$, giving limits 1 and 2.

$L = \int_1^2 \sqrt{u}\,du = \frac{2}{3}(2\sqrt{2} - 1)$.

Answer: $L = \frac{2}{3}(2\sqrt{2} - 1)$

10. A robot follows $\mathbf{r} = (\cos t, \sin t), 0 \le t \le 4\pi$. [6 marks]

(a) Find its total distance and displacement. [3 marks]

The speed is 1, so distance is $\int_0^{4\pi} 1\,dt = 4\pi$.

Both endpoint positions are $(1, 0)$, so displacement is zero.

Answer: Distance 4π; displacement (0, 0).

(b) Explain what is lost if the journey is recorded only as x squared+y squared=1. [3 marks]

The Cartesian equation gives only the set of points.

It does not retain the two circuits, counterclockwise direction or timing.

Answer: The number of circuits, direction and timing are lost.