Vector calculus - Test 3

10 questions | 60 marks | 75 minutes

Polynomial, Bezier and vector applications.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = 2\cos t, y = \sin t$, $0 \le t \le \pi$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$x^2/4 + y^2 = \cos^2 t + \sin^2 t = 1$.

On $0 \le t \le \pi$, $y \ge 0$, selecting the upper half.

Answer: $x^2/4 + y^2 = 1$, $y \ge 0$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

At $0, \pi/2, \pi$, the positions are $(2, 0), (0, 1), (-2, 0)$.

The upper half is traversed once counterclockwise.

Answer: From (2, 0) through (0, 1) to (-2, 0), counterclockwise.

2. A particle moves uniformly from (1, 2, 3) m to (7, -2, 3) m in 2 s. [6 marks]

(a) Write its position function. [3 marks]

The displacement is $(6, -4, 0)$ m, so velocity is $(3, -2, 0)$ m/s.

Add time times velocity to $(1, 2, 3)$.

Answer: $\mathbf{r} = (1, 2, 3) + t(3, -2, 0)$ m.

(b) Find speed and total distance over the two seconds. [3 marks]

Speed is $\sqrt{3^2 + (-2)^2} = \sqrt{13}$ m/s.

The distance in 2 s is $2\sqrt{13}$ m.

Answer: $\sqrt{13}$ m/s and $2\sqrt{13}$ m.

3. $\mathbf{r} = (t^2 - 2t, t^3 - 3t)$, $0 \le t \le 2$. [6 marks]

(a) Find when the particle is at rest and its position. [3 marks]

$\mathbf{v} = (2t - 2, 3t^2 - 3)$.

The first component is zero only at 1; both vanish there, and $\mathbf{r}(1) = (-1, -2)$.

Answer: Rest at t=1, position (-1, -2).

(b) Find its acceleration at that instant. [3 marks]

$\mathbf{a} = (2, 6t)$, giving $(2, 6)$ at 1.

Answer: Acceleration there is (2, 6), not zero.

4. A quadratic Bézier path uses (0, 0), (2, 3), (4, 0). Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Expand its coordinate rules. [3 marks]

$x = 2t(1 - t)(2) + 4t^2 = 4t$.

$y = 2t(1 - t)(3) = 6t - 6t^2$.

Answer: $x = 4t, y = 6t - 6t^2$.

(b) Find the greatest y and explain whether the path passes through the middle control. [3 marks]

$dy/dt = 6 - 12t = 0$ gives $t = 1/2$, and $y = 1.5$.

The point $(2, 3)$ lies above the maximum and is not on the path.

Answer: Maximum 1.5 at t=1/2; it does not reach (2, 3).

5. $\mathbf{v} = (e^t, \cos t), \mathbf{r}(0) = (2, 0)$. [6 marks]

(a) Find the position function. [3 marks]

Integration gives $(e^t + C_1, \sin t + C_2)$.

At zero, $(1 + C_1, C_2) = (2, 0)$, hence $C_1 = 1, C_2 = 0$.

Answer: $\mathbf{r} = (e^t + 1, \sin t)$

(b) Find displacement from 0 to $\pi/2$. [3 marks]

$\mathbf{r}(\pi/2) = (e^{\pi/2} + 1, 1)$ and $\mathbf{r}(0) = (2, 0)$.

Subtraction gives $(e^{\pi/2} - 1, 1)$.

Answer: Displacement from 0 to $\pi/2$ is $(e^{\pi/2} - 1, 1)$.

6. $\mathbf{r} = (e^t, e^{-t})$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (e^t, -e^{-t})$ and $\mathbf{a} = (e^t, e^{-t})$.

Answer: $\mathbf{v} = (e^t, -e^{-t}), \mathbf{a} = (e^t, e^{-t})$

(b) Find speed and speed rate at 0. [3 marks]

At zero, $\mathbf{v} = (1, -1)$ and $\mathbf{a} = (1, 1)$.

Speed is $\sqrt{2}$ and speed rate is $(1 - 1)/\sqrt{2} = 0$.

Answer: At 0 speed $\sqrt{2}$ and speed rate 0.

7. For $x = \cos t, y = \sin t$, t=0. [6 marks]

(a) Find the point and tangent direction at $t = 0$. [3 marks]

$\mathbf{r}(0) = (1, 0)$ and $\mathbf{r}'(0) = (0, 1)$.

The tangent direction is vertical.

Answer: Point (1, 0); vertical direction (0, 1).

(b) Give the tangent line. [3 marks]

The vertical line through $(1, 0)$ is $x = 1$.

Answer: $x = 1$.

8. $\mathbf{r} = (1 + 4\cos 2t,\ -2 - 4\sin 2t)$. [6 marks]

(a) Describe the locus and find its period. [3 marks]

The centre is $(1,\ -2)$ and radius is 4.

Angular speed magnitude 2 gives period π.

Answer: Circle centre (1, -2), radius 4; period π.

(b) Find direction and speed. [3 marks]

At 0 the point is $(5,\ -2)$ and velocity is $(0,\ -8)$, so motion is clockwise.

Speed is $4(2) = 8$.

Answer: Clockwise; speed 8.

9. $\mathbf{r} = (\cos t, \sin t, 2t), 0 \le t \le 2\pi$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{v} = (-\sin t, \cos t, 2)$, so speed is $\sqrt{5}$.

$L = \int_0^{2\pi} \sqrt{5}\, dt$.

Answer: $L = \int_0^{2\pi} \sqrt{5}\, dt$

(b) Evaluate the exact curve length. [3 marks]

$L = 2\pi\sqrt{5}$.

Answer: $L = 2\pi\sqrt{5}$

10. A drawing tool follows a quadratic Bézier curve with controls (0, 0), (2, 2), (4, 0). Use the standard Bézier parameter interval $0 \le t \le 1$. [6 marks]

(a) Expand its coordinate rules and find the highest point. [3 marks]

The quadratic weights give $x = 4t$ and $y = 4t - 4t^2$.

The maximum occurs at $t = 1/2$, yielding $(2, 1)$.

Answer: $x = 4t, y = 4t - 4t^2$; highest (2, 1).

(b) Find the speed at t=1/2 and state whether the tool stops. [3 marks]

$\mathbf{v} = (4, 4 - 8t)$, so at 1/2 it is $(4, 0)$.

The speed is 4, which is nonzero.

Answer: Speed 4; it does not stop.