Vector calculus - Test 4

10 questions | 60 marks | 75 minutes

Tangent directions and three-dimensional travel.

Equivalent correct methods earn the corresponding marks.

1. Consider the parametric curve $x = 3 - t, y = (t-1)^2$, $0 \le t \le 3$. [6 marks]

(a) Eliminate the parameter and state the traced portion. [3 marks]

$t = 3 - x$, so $y = (2-x)^2$.

$0 \le t \le 3$ gives $0 \le x \le 3$.

Answer: $y = (2-x)^2$, $0 \le x \le 3$

(b) Describe its orientation using endpoints and any repeated tracing. [3 marks]

At $t = 0, 1, 3$, the points are $(3, 1), (2, 0), (0, 4)$.

$dx/dt = -1$, so the curve is traversed once from right to left.

Answer: From (3, 1) through (2, 0) to (0, 4), right to left.

2. A runner has position $(2t, 0)$ m and a cyclist starting 10 m behind has position $(-10 + 4t, 0)$ m, for $t \ge 0$. [6 marks]

(a) Find catch-up time and position. [3 marks]

$2t = -10 + 4t$ gives $t = 5$ s.

Both positions are then $(10, 0)$ m.

Answer: 5 s at (10, 0) m.

(b) Explain why equal position at that instant does not mean equal velocity. [3 marks]

The constant derivatives are $(2, 0)$ and $(4, 0)$ m/s.

Equal values of position at one time do not require equal derivatives.

Answer: The runner velocity is (2, 0), cyclist velocity (4, 0).

3. $\mathbf{r} = (t^2, 4t)$, $0 \le t \le 2$. [6 marks]

(a) Eliminate t and state the restriction. [3 marks]

$t = y/4$, giving $x = y^2/16$.

The interval gives $0 \le y \le 8$.

Answer: $x = y^2/16$, $0 \le y \le 8$.

(b) Find velocity and decide whether rest is possible. [3 marks]

$\mathbf{v} = (2t, 4)$.

The second component never vanishes, so rest is impossible.

Answer: Velocity (2 t, 4); the particle never stops.

4. A quadratic path ends with controls P 1=(1, 2), P 2=(3, 1). A second quadratic begins with Q 0=P 2 and unknown Q 1. [6 marks]

(a) Find the first path final derivative. [3 marks]

$\mathbf{B}'(1) = 2[(3, 1) - (1, 2)] = (4, -2)$.

Answer: $(4, -2)$

(b) Find Q 1 so the two derivatives are equal at their join. [3 marks]

The second initial derivative is $2[Q_1 - (3, 1)]$.

Equating this to $(4, -2)$ gives $Q_1 = (3, 1) + (2, -1) = (5, 0)$.

Answer: $Q_1 = (5, 0)$

5. $\mathbf{v} = (3t^2, 2t, 1)$,$\mathbf{r}(0) = (0, 1, -1)$. [6 marks]

(a) Find the position function. [3 marks]

Integrating gives $(t^3 + C_1, t^2 + C_2, t + C_3)$.

The initial position gives $(C_1, C_2, C_3) = (0, 1, -1)$.

Answer: $\mathbf{r} = (t^3, t^2 + 1, t - 1)$

(b) Find displacement from 0 to 1. [3 marks]

$\mathbf{r}(1) = (1, 2, 0)$ and $\mathbf{r}(0) = (0, 1, -1)$.

The displacement is $(1, 1, 1)$.

Answer: Displacement from 0 to 1 is (1, 1, 1).

6. $\mathbf{r} = (3t, 4t, 12t)$. [6 marks]

(a) Find velocity and acceleration. [3 marks]

$\mathbf{v} = (3, 4, 12)$ and $\mathbf{a} = (0, 0, 0)$.

Answer: $\mathbf{v} = (3, 4, 12), \mathbf{a} = (0, 0, 0)$

(b) Find speed and explain its constancy. [3 marks]

Speed is $\sqrt{9 + 16 + 144} = 13$.

Because velocity is a constant vector, its magnitude is constant.

Answer: Speed 13, constant.

7. For $\mathbf{r} = (t, t^2, t^3)$, t=1. [6 marks]

(a) Find the point and derivative vector at $t = 1$. [3 marks]

$\mathbf{r}(1) = (1, 1, 1)$ and $\mathbf{r}'(1) = (1, 2, 3)$.

Answer: Point (1, 1, 1); direction (1, 2, 3).

(b) Give a vector equation of the tangent line, using a new parameter. [3 marks]

A fresh line parameter gives $\mathbf{x} = (1, 1, 1) + s(1, 2, 3)$.

Answer: $\mathbf{x} = (1, 1, 1) + s(1, 2, 3)$.

8. $\mathbf{r} = (3\cos t, 3\sin t, 4t)$. [6 marks]

(a) Describe the helix radius and find the time for one planar turn. Does the full position function have a period? [3 marks]

The planar coordinates trace a radius-3 circle in 2π time units.

The z coordinate rises uniformly, creating a helix; the full position does not repeat.

Answer: Helix of radius 3; one planar turn takes 2π. The full position has no period.

(b) Find speed and rise per turn. [3 marks]

Speed is $\sqrt{3^2 + 4^2} = 5$.

In one planar turn, z changes by $4(2\pi) = 8\pi$.

Answer: Speed 5; rise per turn 8π.

9. $\mathbf{r} = (4\cos t, 4\sin t), 0 \le t \le \pi/2$. [6 marks]

(a) Set up the arc-length integral. [3 marks]

$\mathbf{v} = (-4\sin t, 4\cos t)$, so speed is 4.

$L = \int_0^{\pi/2} 4\, dt$.

Answer: $L = \int_0^{\pi/2} 4dt$

(b) Evaluate the exact curve length. [3 marks]

$L = 4(\pi/2) = 2\pi$.

Answer: $L = 2\pi$

10. A helix is $\mathbf{r} = (2\cos t, 2\sin t, 3t)$, from 0 to π. [6 marks]

(a) Find the endpoint separation. [3 marks]

The endpoints are $(2, 0, 0)$ and $(-2, 0, 3\pi)$.

Their difference is $(-4, 0, 3\pi)$, with magnitude $\sqrt{16 + 9\pi^2}$.

Answer: $\sqrt{16 + 9\pi^2}$

(b) Find path length and explain why it exceeds the separation. [3 marks]

Speed is $\sqrt{4 + 9} = \sqrt{13}$, so length is $\pi\sqrt{13}$.

The curved helical journey exceeds the straight segment between its endpoints.

Answer: $\pi\sqrt{13}$; the path is curved.